Errata for Higher categories and Homotopical Algebra
(September 15, 2023)

The online version of the text is systematically corrected (without changing the numbering of theorems). Here is the list of corrections so far, with references to the printed book.

- Page 10, Example 1.3.2. Δ^n should be $[n]$.
- Page 45, end of the proof of Proposition 2.3.11. The map from T'_1 to Y' is not (f_1, p_0) but rather (f_1, yp_0).
- Page 69, second part of the proof of Proposition 2.5.3. The map $p : X \to Y$ should be $f : X \to Y$.
- Page 71, proof of Proposition 2.5.7. The map $h(\partial_1 \otimes 1_X)$ should be $h(\partial_1 \otimes 1_Y)$.
- Page 78, proof of Proposition 3.1.13. The case where $F_i \cap F_j$ is empty does not occur for $n \neq 1$ (and that is an essential part of the proof).
- Page 160, Theorem 4.4.14. There is a typo: a morphism between fibrants objects over C is a fibration if and only if it is a left fibration.
- Page 169, Paragraph 4.4.35. In Diagram (4.4.35.1), we should label the map $A' \to A/b_1$ as $u : A' \to A/b_1$. At the bottom of the page, just after Diagram (4.4.35.4), we should correct the text as follows.

In particular, there is a canonical monomorphism

$$a' \backslash A_{b_1} \to (v, a_0) \backslash A_{b_1} \tag{4.4.35.5}$$

which can be identified with the inclusion

$$A_{b_1} \times_{(v,b_0)\backslash(A/b_1)} (v, a_0)\backslash(A/b_1) \subset A_{b_1} \times_{A/b_1} (v, a_0)\backslash(A/b_1)\,.$$

In the case where p is a an inner fibration between ∞-categories, one can show that the map (4.4.35.5) is an equivalence of ∞-categories as follows, up to equivalences of ∞-categories (due to Proposition 4.2.9, it corresponds to the identification

$$A_{b_1} \times_{(v,b_0)\backslash\backslash(A//b_1)} (v, a_0)\backslash\backslash(A//b_1) = A_{b_1} \times_{A//b_1} (v, a_0)\backslash\backslash(A//b_1)\,.$$

- Page 170, in the proof of Theorem 4.4.36, the sentence "The isomorphism (4.4.35.5) thus proves that conditions (ii) and (iii) are equivalent." should be replaced by "The equivalence (4.4.35.5) thus proves that conditions (ii) and (iii) are equivalent.".

• Page 196–197, Definition 5.2.3 is slightly flawed and does not fit perfectly with the description of the universe used subsequently in the book. Here is the fully correct formulation.

Definition 5.2.3 We fix a Grothendieck universe $\mathbf{U}$. A set is $\mathbf{U}$-*small* is it belongs to $\mathbf{U}$. One defines $\mathbf{\Delta}$ so that its set of arrows is $\mathbf{U}$-small. We define the simplicial set U of morphisms of simplicial sets with $\mathbf{U}$-small fibres as follows. An element of U_n is a map $p : X \to \Delta^n$, such that X takes its values in $\mathbf{U}$-small sets, together with a choice, for any map $f : \Delta^m \to \Delta^n$, of a Cartesian square of $\mathbf{U}$-small simplicial sets of the following form.

$$\begin{array}{ccc} f^*(X) & \xrightarrow{\tilde{f}} & X \\ \Big\downarrow{\scriptstyle f^*p} & & \Big\downarrow{\scriptstyle p} \\ \Delta^m & \xrightarrow{f} & \Delta^n \end{array}$$

with the constraint that $1^*_{\Delta^n}(X) = X$ and $\tilde{1}_{\Delta^n} = 1_X$.

One defines the simplicial set $\mathcal{S}$ of *left fibrations with specified* $\mathbf{U}$-*small fibers* as the sub-object of U whose elements correspond to left fibrations of codomain Δ^n with specified pull-back squares of $\mathbf{U}$-small simplicial sets as above.

One checks immediately that $\mathcal{S}^{op}$ can be interpreted as the simplicial set of *right fibrations with specified* $\mathbf{U}$-*small fibers*, i.e., is canonically isomorphic to the sub-object of U whose elements are the right fibrations of codomain Δ^n with suitably specified pull-back squares.

There is a pointed version of U, which we denote by $U_\bullet$. A map $\Delta^n \to U_\bullet$ is a map $p : X \to \Delta^n$ with X a presheaf of $\mathbf{U}$-small sets, equipped with pull-backs as above, together with a section $s : \Delta^n \to X$ of p. Forgetting the section s defines a morphism of simplicial sets

$$\pi : U_\bullet \to U .$$

One defines similarly

$$p_{univ} : \mathcal{S}_\bullet \to \mathcal{S}$$

as the pull-back of $\pi : U_\bullet \to U$ along the inclusion $\mathcal{S} \subset U$.

• Page 200, in order to avoid set–theoretic problems, we should add the following sentences right at the begining of the proof of Lemma 5.2.12.

There is a simplicial subset $\mathcal{K} \subset \mathcal{S}$ such that a map $F : A \to \mathcal{S}$ factors through $\mathcal{K}$ if and only if F classifies a Kan fibration. We want to prove that $\mathcal{K}$ is a Kan complex. That means that it is sufficient to prove this lemma in the case where the anodyne extension $K \to L$ is a horn inclusion. Therefore, we may assume, without loss of generality, that L itself is $\mathbf{U}$-small.

• Page 203, Remark 5.2.15 is flawed and should be replaced as follows.

Remark 5.2.15 An inspection of the proof of Theorems 5.2.10 and 5.2.14 shows that a monomorphism of small simplicial sets $i : A \to B$ has the left lifting property with respect to $\mathcal{S} \to \Delta^0$ (for all universes $\mathbf{U}$) whenever the induced functor $\mathbf{L}i_! : RFib(A) \to RFib(B)$ is fully faithful.

• Pages 207–208, Proposition 5.3.8 and its proof should be reformulated as follows (this proposition is only used in the proof of 5.3.11, and this does not affect the latter).

Proposition 5.3.8 *If the morphisms $p : X \to A$ and $q : Y \to X$ are left fibrations, then the morphism $\pi_{X,Y} : \underline{\mathrm{Hom}}_A(X, Y) \to A$ is a fibration of the Joyal model category structure.*

Proof The functor $(Y, q) \mapsto (\underline{\mathrm{Hom}}_A(X, Y), \pi_{X,Y})$ is right adjoint to the functor $(-)\times_A X$. The latter preserves monomorphisms, and, by virtue of Proposition 5.3.5, it also preserves the class of weak equivalences of the model category structure on $sSet/A$ induced by the Joyal model category structure. In particular, we have here a Quillen pair. □

• Page 221, in the proof of fullness, there is a size issue which must be address. The following paragraph should be inserted.

Let us prove the property of fullness. Let $p : X \to A$ and $q : Y \to A$ be left fibrations classified by F and G, respectively, and let $\psi : X \to Y$ be a morphism of simplicial sets over A. By virtue of Proposition 5.4.3, the map ψ corresponds to a left fibration $\pi : W \to \Delta^1 \times A$, such that the fibers of π at 0 and 1 are homotopic over A to X and Y, respectively. We claim, that, since X and Y have $\mathbf{U}$-small fibers, we may choose W with the same property. This is proved as follows. We observe that this is obvious whenever A itself is $\mathbf{U}$-small (since the model structures involved in Proposition 5.4.3 may be restricted to $\mathbf{U}$-small objects). In general, we may assume that π is a minimal left fibration. We observe furthermore that $\Delta^1 \times A$ is a filtered union of subobjects of the form $\Delta^1 \times B$, where B runs over $\mathbf{U}$-small subobjects of A. It is thus sufficient to prove that the domain of the pullback of π over such $\Delta^1 \times B$ is $\mathbf{U}$-small. By minimality, it is sufficient to prove that such a pullback is fiberwise equivalent to a left fibration with $\mathbf{U}$-small fibers, which we already know. Using Lemma 5.4.4, one can find a morphism $\Delta^1 \to \underline{\mathrm{Hom}}(A, \mathcal{S})$ classifying the left fibration $\pi : W \to \Delta^1 \times A$, out of which [...]

• Page 228, at the end of the proof of Theorem 5.5.7, a false computation is used to conclude that the co-unit map $\delta_!\delta^*(X) \to X$ is a weak equivalence. Between the sentences "Similarly, the class of bisimplicial sets X such that the co-unit map $\delta_!\delta^*(X) \to X$ is a weak equivalence is saturated by monomorphisms." and "In other words, the adjoint pair $(\delta_!, \delta^*)$ is a Quillen equivalence." the text should be replaced by the following words:

In the case where $X = \Delta^m \boxtimes \Delta^n$ is representable, we have $\delta^*(X) = \Delta^m \times \Delta^n$ weakly contractible, and therefore, since $\delta_!$ preserves the terminal object as well as weak equivalences, by virtue of Corollary 1.3.10, to prove that the co-unit map $\delta_!\delta^*(X) \to X$ is a weak equivalence for all X, we only have to check that $\Delta^m \times \Delta^n$ is weakly contractible for all m and n, which is an easy exercise.

• Page 232, there are a couple of typos at the end of the proof of Lemma 5.5.16: some T's have been replaced by Y's. One should read:
Therefore, the map $X^K \to T^K$ is a trivial fibration for all K. This implies that the map $X^L \to X^K \times_{T^K} T^L$ is a trivial fibration for any monomorphism $K \to L$. Therefore, the map $q : X \to T$ is a trivial fibration of bisimplicial sets, and, since i is a monomorphism, it is a retract of j, which is in particular a left (right) bi-anodyne extension.

• Page 285, the proof of Corollary 6.3.5 is correct only for C cocomplete. For the general case, we proceed as follows. Let $F : I \to C$ a functor which has a limit in C. Let $\mathbf{U}$ be a universe such that both I and C are $\mathbf{U}$-small. We will first prove that $h_C(\varprojlim_i F_i)$ is a limit of $h_C(F)$ in the ∞-category of presheaves of $\mathbf{U}$-small ∞-groupoids. For all objects X of C, we have by the Yoneda Lemma canonical invertible maps

$$\begin{aligned} \operatorname{Hom}(X, \varprojlim_i F_i) &\simeq \operatorname{Hom}(X_I, F) \\ &\simeq \operatorname{Hom}(h_C(X_I), h_C(F)) \\ &= \operatorname{Hom}(h_C(X)_I, h_C(F)) \end{aligned}$$

functorially in X and F, where X_I denotes the constant diagram indexed by I with value X. This means that, in the identification above, we may take X to be a diagram in C indexed by some $\mathbf{U}$-small ∞-category J (and the Hom's as those of the category of functors from J to C or to $\widehat{C}$). In other words, for such diagram, since the functor $Y \mapsto Y_I$ commutes with colimits (by Corollary

6.2.10) we also have:

$$\begin{aligned}\mathrm{Hom}(\varinjlim_j h_C(X_j), h_C(\varprojlim_i F_i)) &\simeq \mathrm{Hom}(X, (\varprojlim_i F_i)_J)\\ &\simeq \mathrm{Hom}(X_I, F_J)\\ &\simeq \mathrm{Hom}(h_C(X)_I, h_C(F)_J)\\ &\simeq \mathrm{Hom}((\varinjlim_j h_C(X_j))_I, h_C(F))\,.\end{aligned}$$

Since any presheaf is a small colimit of representable presheaves (Corollary 6.2.16), this shows that, for any presheaf Φ on C with values in $\mathcal{S}$, we have:

$$\mathrm{Hom}(\Phi, h_C(\varprojlim_i F_i)) \simeq \mathrm{Hom}(\Phi_I, h_C(F))\,.$$

In other words, there is a canonical invertible map

$$h_C(\varprojlim_i F_i) \to \varprojlim_i h_C(F_i)\,.$$

Evaluating at an object X of C and using Corollary 6.2.10 finishes the proof.

• Page 288, the first sentence of the proof of Theorem 6.3.13 should read:

Proof Proposition 6.3.9 gives the essential surjectivity of the functor h_A^*.

• Page 319, there is a typo (which is repeated several times in the proof of Proposition 7.1.3). In diagram (7.1.1.1), the object at the lower left corner should be $\underline{\mathrm{Hom}}_W(W, D)$:

(7.1.1.1)
$$\begin{array}{ccc}\underline{\mathrm{Hom}}_W(C,D) & \longrightarrow & \underline{\mathrm{Hom}}(C,D)\\ \downarrow & & \downarrow\\ \underline{\mathrm{Hom}}_W(W,D) & \longrightarrow & \underline{\mathrm{Hom}}(W,D)\end{array}$$

The proof of Proposition 7.1.3 should be:

Proof For any ∞-category X, the functor $h(-, X)$ takes anodyne extensions to trivial fibrations; see Corollary 3.5.13. Therefore, if we put $W' = Ex^\infty(W)$, we have a trivial fibration

$$\underline{\mathrm{Hom}}(W', D) = h(W', D) \to h(W, D) = \underline{\mathrm{Hom}}_W(W, D)\,.$$

We define C' by forming the following push-out square.

$$
\begin{tikzcd}
W \arrow[r] \arrow[d] & C \arrow[d] \\
W' \arrow[r] & C'
\end{tikzcd}
\tag{7.1.3.1}
$$

Since $\underline{\mathrm{Hom}}(W', D) = h(W', D)$, we have $\underline{\mathrm{Hom}}_{W'}(C', D) = \underline{\mathrm{Hom}}(C', D)$, and therefore, the Cartesian square obtained by applying the functor $\underline{\mathrm{Hom}}(-, D)$ to the coCartesian square (7.1.3.1), together with the Cartesian square (7.1.1.1), give a Cartesian square of the following form.

$$
\begin{tikzcd}
\underline{\mathrm{Hom}}(C', D) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_W(C, D) \arrow[d] \\
\underline{\mathrm{Hom}}(W', D) \arrow[r] & h(W, D)
\end{tikzcd}
\tag{7.1.3.2}
$$

If we choose a fibrant resolution $W^{-1}C$ of C' in the Joyal model category structure, we get a trivial fibration of the form

$$\underline{\mathrm{Hom}}(W^{-1}C, D) \to \underline{\mathrm{Hom}}(C', D)\,.$$

On the other hand, since the lower horizontal map of diagram (7.1.3.2) is a trivial fibration, so is the upper one, hence the inclusion $\gamma : C \to W^{-1}C$ is a localisation of C by W. By definition, the map γ exhibits $W^{-1}C$ as a representation of the functor $\pi_0(k(\underline{\mathrm{Hom}}_W(C, -)))$ in the homotopy category of the Joyal model category structure. The Yoneda Lemma thus implies that the pair $(W^{-1}C, \gamma)$ is unique up to a unique isomorphism in the homotopy category of the Joyal model category structure. □

• Page 336–339, as spotted by Simon Henry, Theorem 7.2.25 is not correct as stated (there are missing conditions) and the proof should be corrected. The statement should be:

Theorem 7.2.25 *Let us assume that there is a class of trivial fibrations with respect to W. We fix a class F of W-local maps and we denote, for each object x of C, by $C(x)$ the full subcategory of the slice C/x with objects the maps $p : y \to x$ that belong to F. We also assume the following properties:*

(i) any trivial fibration belongs to F;

(ii) for any map $p : x \to y$ in F and any trivial fibration $q : y \to z$, the map $qp : x \to z$ is in F;

(iii) for any object x, the full subcategory $W(x)$ of $C(x)$ spanned by trivial fibrations of codomain x forms a right calculus of fractions at $(x, 1_x)$ in

$C(x)$ with respect to the class of maps whose image through the forgetful functor $C(x) \subset C/x \to C$ belong to W.

Then, for any Cartesian square in C of the form

(7.2.25.1)
$$\begin{tikzcd} s \arrow[r, "f'"] \arrow[d, "q"] & y' \arrow[d, "p"] \\ t \arrow[r, "f"] & y \end{tikzcd}$$

in which the map p is in F, the square

(7.2.25.2)
$$\begin{tikzcd} \gamma(s) \arrow[r, "\gamma(f')"] \arrow[d, "\gamma(q)"] & \gamma(y') \arrow[d, "\gamma(p)"] \\ \gamma(t) \arrow[r, "\gamma(f)"] & \gamma(y) \end{tikzcd}$$

is Cartesian in $W^{-1}C$.

Proof By Remark 6.6.11, the image of the Cartesian square (7.2.25.1) by the Yoneda embedding of C may be realized as a Cartesian square of the contravariant model category structure over C of the form

(7.2.25.3)
$$\begin{tikzcd} C_{/s} \arrow[r, "f'_!"] \arrow[d, "q_!"] & C_{/y'} \arrow[d, "p_!"] \\ C_{/t} \arrow[r, "f_!"] & C_{/y} \end{tikzcd}$$

in which all maps are right fibrations (and each structural map of the form $C_{/c} \to C$ is a right fibration whose domain has a final object which is sent to c in C). Each object x of C, will be equipped with the right calculus of fractions $W(x)$ associated to some choice of a class of trivial fibrations, i.e., which consists of the full subcategory of C/x whose objects are the pairs (z, s), with $s : z \to x$ a trivial fibration. Pulling-back along the map $\pi(x) : W(x) \to C$, we thus get a Cartesian square over $W(x)$ of the form

(7.2.25.4)
$$\begin{tikzcd} W(x)_{/s} \arrow[r, "f'_!"] \arrow[d, "q_!"] & W(x)_{/y'} \arrow[d, "p_!"] \\ W(x)_{/t} \arrow[r, "f_!"] & W(x)_{/y} \end{tikzcd}$$

(in which we have put $W(x)_{/c} = C_{/c} \times_C W(x)$). By virtue of the interpretation of Theorem 7.2.8 made in Remark 7.2.10, it is sufficient to prove that diagram (7.2.25.4) is homotopy Cartesian in the Kan-Quillen model category structure.

For this purpose, we shall apply the dual version of Proposition 4.6.11 to the functor $p_! : W(x)_{/y'} \to W(x)_{/y}$. In other words, we have to give ourselves a map $w : a_0 \to a_1$ in $W(x)_{/y}$ and see that pulling back along the induced functor

$$w_! : (W(x)_{/y})/a_0 \to (W(x)_{/y})/a_1 \tag{7.2.25.5}$$

gives a weak homotopy equivalence of the form

$$w_! : (W(x)_{/y'})/a_0 \to (W(x)_{/y'})/a_1\,. \tag{7.2.25.6}$$

Such a map w corresponds essentially to a commutative diagram in C, of the form

$$\begin{tikzcd} z_0 \arrow[r, "g_0"] \arrow[d, "s_0"] \arrow[dr, "w"] & y \\ x & z_1 \arrow[l, "s_1"] \arrow[u, "g_1"'] \end{tikzcd} \tag{7.2.25.7}$$

such that s_0 and s_1 are trivial fibrations. We form the following Cartesian squares.

$$\begin{tikzcd} z'_0 \arrow[r, "w'"] \arrow[d, "p_0"] & z'_1 \arrow[r, "g'_1"] \arrow[d, "p_1"] & y' \arrow[d, "p"] \\ z_0 \arrow[r, "w"] & z_1 \arrow[r, "g_1"] & y \end{tikzcd}$$

Using right caclculus of fractions at $(x, 1_x)$ in $C(x)$ (under the form of equivalence (7.2.10.3)), we observe that, for each $i = 0, 1$, the ∞-category $(W(x)_{/y'})/a_i$ has the weak homotopy type of the mapping space of maps from $(x, 1_x)$ to $(z'_i, s_i p_i)$ in the localization of $C(x)$. In other words, it suffices to know that w' induces an isomorphism in the localization $C(x)$. But, since p is W-local, the map w' is a weak equivalence. □

• Page 368, 5th line of the proof of Proposition 7.5.6, to make the link with Theorem 7.2.25 clearer, one should add the following:
Moreover, since each slice C/x is an ∞-category with weak equivalences and fibrations as well, this does not apply to C only but to each slice C/x as well. Therefore, by virtue of Proposition 7.4.16, Theorem 7.2.25 applies then here (with F the class of fibrations in C_f, so that each ∞-category $C(x)$ simply is the full subcategory of C/x spanned by fibrations of codomain x).

• Page 343, there are typos in the definition of a categorical realisation of a décalage: in conditions (iii) and (iv), the functor i should be applied to the diagrams (in examples, the functor i is an inclusion functor, so that this mistake does not affect applications in the book). We should read:

(iii) For any map $f : a \to b$ in A, the commutative square

$$
\begin{tikzcd}
i(a) \arrow[r, "i(\eta_a)"] \arrow[d, "i(f)"'] & i(D(a)) \arrow[d, "i(D(f))"] \\
i(b) \arrow[r, "i(\eta_b)"] & i(D(b))
\end{tikzcd}
\tag{7.3.7.2}
$$

is Cartesian.

(iv) For any object a of A, the commutative square

$$
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & i(\omega) \arrow[d, "i(\pi_a)"] \\
i(a) \arrow[r, "i(\eta_a)"] & i(D(a))
\end{tikzcd}
\tag{7.3.7.3}
$$

is Cartesian.

• Page 357, in Definition 7.4.6, there is an oversight: the map $f : x \to y$ must be in F.