Errata for Higher categories and Homotopical Algebra
(April 20, 2022)

The online version of the text is systematically corrected (without changing the numbering of theorems). Here is the list of corrections so far, with references to the printed book.

• Page 169, Paragraph 4.4.35. In Diagram (4.4.35.1), we should label the map $A' \to A/b_1$ as $u : A' \to A/b_1$. At the bottom of the page, just after Diagram (4.4.35.4), we should correct the text as follows.

In particular, there is a canonical monomorphism

$$a'\backslash A_{b_1} \to (v, a_0)\backslash A_{b_1} \tag{4.4.35.5}$$

which can be identified with the inclusion

$$A_{b_1} \times_{(v,b_0)\backslash(A/b_1)} (v, a_0)\backslash(A/b_1) \subset A_{b_1} \times_{A/b_1} (v, a_0)\backslash(A/b_1)\,.$$

In the case where p is a an isofibration between ∞-categories, one can show that the map (4.4.35.5) is an equivalence of ∞-categories (we leave this as an exercise to the reader: one may for instance check that the map induces a bijection on sets of isomorphism classes, and then use the fact the formation of the map (4.4.35.5) is compatible with the functor $\underline{\mathrm{Hom}}(X, -)$ up to equivalences of ∞-categories).

• Page 170, in the proof of Theorem 4.4.36, the sentence "The isomorphism (4.4.35.5) thus proves that conditions (ii) and (iii) are equivalent." should be replaced by "The equivalence (4.4.35.5) thus proves that conditions (ii) and (iii) are equivalent.".

• Page 196–197, Definition 5.2.3 is slightly flawed and does not fit perfectly with the description of the universe used subsequently in the book. Here is the fully correct formulation.

Definition 5.2.3 We fix a Grothendieck universe $\mathbf{U}$. A set is $\mathbf{U}$-*small* is it belongs to $\mathbf{U}$. One defines $\mathbf{\Delta}$ so that its set of arrows is $\mathbf{U}$-small. We define the simplicial set U of morphisms of simplicial sets with $\mathbf{U}$-small fibres as follows. An element of U_n is a map $p : X \to \Delta^n$, such that X takes its values in $\mathbf{U}$-small sets, together with a choice, for any map $f : \Delta^m \to \Delta^n$, of a Cartesian square of $\mathbf{U}$-small simplicial sets of the following form.

$$\begin{array}{ccc} f^*(X) & \xrightarrow{\tilde{f}} & X \\ \Big\downarrow{\scriptstyle f^*p} & & \Big\downarrow{\scriptstyle p} \\ \Delta^m & \xrightarrow{f} & \Delta^n \end{array}$$

with the constraint that $1^*_{\Delta^n}(X) = X$ and $\tilde{1}_{\Delta^n} = 1_X$.

One defines the simplicial set $\mathcal{S}$ of *left fibrations with specified* $\mathbf{U}$*-small fibers* as the sub-object of U whose elements correspond to left fibrations of codomain Δ^n with specified pull-back squares of $\mathbf{U}$-small simplicial sets as above.

One checks immediately that $\mathcal{S}^{op}$ can be interpreted as the simplicial set of *right fibrations with specified* $\mathbf{U}$*-small fibers*, i.e., is canonically isomorphic to the sub-object of U whose elements are the right fibrations of codomain Δ^n with suitably specified pull-back squares.

There is a pointed version of U, which we denote by $U_\bullet$. A map $\Delta^n \to U_\bullet$ is a map $p : X \to \Delta^n$ with X a presheaf of $\mathbf{U}$-small sets, equipped with pull-backs as above, together with a section $s : \Delta^n \to X$ of p. Forgetting the section s defines a morphism of simplicial sets

$$\pi : U_\bullet \to U \,.$$

One defines similarly

$$p_{univ} : \mathcal{S}_\bullet \to \mathcal{S}$$

as the pull-back of $\pi : U_\bullet \to U$ along the inclusion $\mathcal{S} \subset U$.

• Page 200, in order to avoid set–theoretic problems, we should add the following sentences right at the begining of the proof of Lemma 5.2.12.

There is a simplicial subset $\mathcal{K} \subset \mathcal{S}$ such that a map $F : A \to \mathcal{S}$ factors through $\mathcal{K}$ if and only if F classifies a Kan fibration. We want to prove that $\mathcal{K}$ is a Kan complex. That means that it is sufficient to prove this lemma in the case where the anodyne extension $K \to L$ is a horn inclusion. Therefore, we may assume, without loss of generality, that L itself is $\mathbf{U}$-small.

• Page 203, Remark 5.2.15 is flawed and should be replaced as follows.

Remark 5.2.15 An inspection of the proof of Theorems 5.2.10 and 5.2.14 shows that a monomorphism of small simplicial sets $i : A \to B$ has the left lifting property with respect to $\mathcal{S} \to \Delta^0$ (for all universes $\mathbf{U}$) whenever the induced functor $\mathbf{L}i_! : RFib(A) \to RFib(B)$ is fully faithful.

• Pages 207–208, Proposition 5.3.8 and its proof should be reformulated as follows (this proposition is only used in the proof of 5.3.11, and this does not affect the latter).

Proposition 5.3.8 *If the morphisms $p : X \to A$ and $q : Y \to X$ are left fibrations, then the morphism $\pi_{X,Y} : \underline{\mathrm{Hom}}_A(X, Y) \to A$ is a fibration of the Joyal model category structure.*

Proof The functor $(Y, q) \mapsto (\underline{\mathrm{Hom}}_A(X, Y), \pi_{X,Y})$ is right adjoint to the functor $(-) \times_A X$. The latter preserves monomorphisms, and, by virtue of Proposition 5.3.5, it also preserves the class of weak equivalences of the model category structure on *sSet*$/A$ induced by the Joyal model category structure. In particular, we have here a Quillen pair. □

• Page 221, in the proof of fullness, there is a size issue which must be address. The following paragraph should be inserted.

Let us prove the property of fulness. Let $p : X \to A$ and $q : Y \to A$ be left fibrations classified by F and G, respectively, and let $\psi : X \to Y$ be a morphism of simplicial sets over A. By virtue of Proposition 5.4.3, the map ψ corresponds to a left fibration $\pi : W \to \Delta^1 \times A$, such that the fibers of π at 0 and 1 are homotopic over A to X and Y, respectively. We claim, that, since X and Y have **U**-small fibers, we may choose W with the same property. This is proved as follows. We observe that this is obvious whenever A itself is **U**-small (since the model structures involved in Proposition 5.4.3 may be restricted to **U**-small objects). In general, we may assume that π is a minimal left fibration. We observe furthermore that $\Delta^1 \times A$ is a filtered union of subobjects of the form $\Delta^1 \times B$, where B runs over **U**-small subobjects of A. It is thus sufficient to prove that the domain of the pullback of π over such $\Delta^1 \times B$ is **U**-small. By minimality, it is sufficient to prove that such a pullback is fiberwise equivalent to a left fibration with **U**-small fibers, which we already know. Using Lemma 5.4.4, one can find a morphism $\Delta^1 \to \underline{\mathrm{Hom}}(A, \mathcal{S})$ classifying the left fibration $\pi : W \to \Delta^1 \times A$, out of which [...]

• Page 228, at the end of the proof of Theorem 5.5.7, a false computation is used to conclude that the co-unit map $\delta_!\delta^*(X) \to X$ is a weak equivalence. Between the sentences “Similarly, the class of bisimplicial sets X such that the co-unit map $\delta_!\delta^*(X) \to X$ is a weak equivalence is saturated by monomorphisms.” and “In other words, the adjoint pair $(\delta_!, \delta^*)$ is a Quillen equivalence.” the text should be replaced by the following words:

In the case where $X = \Delta^m \boxtimes \Delta^n$ is representable, we have $\delta^*(X) = \Delta^m \times \Delta^n$ weakly contractible, and therefore, since $\delta_!$ preserves the terminal object as well as weak equivalences, by virtue of Corollary 1.3.10, to prove that the co-unit map $\delta_!\delta^*(X) \to X$ is a weak equivalence for all X, we only have to check that $\Delta^m \times \Delta^n$ is weakly contractible for all m and n, which is an easy exercise.

• Page 232, there are a couple of typos at the end of the proof of Lemma 5.5.16: some T's have been replaced by Y's. One should read:
Therefore, the map $X^K \to T^K$ is a trivial fibration for all K. This implies that the map $X^L \to X^K \times_{T^K} T^L$ is a trivial fibration for any monomorphism $K \to L$. Therefore, the map $q : X \to T$ is a trivial fibration of bisimplicial sets, and, since i is a monomorphism, it is a retract of j, which is in particular a left (right) bi-anodyne extension.

• Page 285, the proof of Corollary 6.3.5 is correct only for C cocomplete. For the general case, we proceed as follows. Let $F : I \to C$ a functor which has a limit in C. Let $\mathbf{U}$ be a universe such that both I and C are $\mathbf{U}$-small. We will first prove that $h_C(\varprojlim_i F_i)$ is a limit of $h_C(F)$ in the ∞-category of presheaves of $\mathbf{U}$-small ∞-groupoids. For all objects X of C, we have by the Yoneda Lemma canonical invertible maps

$$\begin{aligned}\mathrm{Hom}(X, \varprojlim_i F_i) &\simeq \mathrm{Hom}(X_I, F)\\ &\simeq \mathrm{Hom}(h_C(X_I), h_C(F))\\ &= \mathrm{Hom}(h_C(X)_I, h_C(F))\end{aligned}$$

functorially in X and F, where X_I denotes the constant diagram indexed by I with value X. This means that, in the identification above, we may take X to be a diagram in C indexed by some $\mathbf{U}$-small ∞-category J (and the Hom's as those of the category of functors from J to C or to $\widehat{C}$). In other words, for such diagram, since the functor $Y \mapsto Y_I$ commutes with colimits (by Corollary 6.2.10) we also have:

$$\begin{aligned}\mathrm{Hom}(\varinjlim_j h_C(X_j), h_C(\varprojlim_i F_i)) &\simeq \mathrm{Hom}(X, (\varprojlim_i F_i)_J)\\ &\simeq \mathrm{Hom}(X_I, F_J)\\ &\simeq \mathrm{Hom}(h_C(X)_I, h_C(F)_J)\\ &\simeq \mathrm{Hom}((\varinjlim_j h_C(X_j))_I, h_C(F))\,.\end{aligned}$$

Since any presheaf is a small colimit of representable presheaves (Corollary

6.2.16), this shows that, for any presheaf Φ on C with values in $\mathcal{S}$, we have:

$$\mathrm{Hom}(\Phi, h_C(\varprojlim_i F_i)) \simeq \mathrm{Hom}(\Phi_I, h_C(F)) .$$

In other words, there is a canonical invertible map

$$h_C(\varprojlim_i F_i) \to \varprojlim_i h_C(F_i) .$$

Evaluating at an object X of C and using Corollary 6.2.10 finishes the proof.

• Page 288, the first sentence of the proof of Theorem 6.3.13 should read:

Proof Proposition 6.3.9 gives the essential surjectivity of the functor h_A^*.

• Page 319, there is a typo (which is repeated several times in the proof of Proposition 7.1.3). In diagram (7.1.1.1), the object at the lower left corner should be $\underline{\mathrm{Hom}}_W(W, D)$:

(7.1.1.1)
$$\begin{tikzcd}
\underline{\mathrm{Hom}}_W(C, D) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(C, D) \arrow[d] \\
\underline{\mathrm{Hom}}_W(W, D) \arrow[r] & \underline{\mathrm{Hom}}(W, D)
\end{tikzcd}$$

The proof of Proposition 7.1.3 should be:

Proof For any ∞-category X, the functor $h(-, X)$ takes anodyne extensions to trivial fibrations; see Corollary 3.5.13. Therefore, if we put $W' = Ex^\infty(W)$, we have a trivial fibration

$$\underline{\mathrm{Hom}}(W', D) = h(W', D) \to h(W, D) = \underline{\mathrm{Hom}}_W(W, D) .$$

We define C' by forming the following push-out square.

(7.1.3.1)
$$\begin{tikzcd}
W \arrow[r] \arrow[d] & C \arrow[d] \\
W' \arrow[r] & C'
\end{tikzcd}$$

Since $\underline{\mathrm{Hom}}(W', D) = h(W', D)$, we have $\underline{\mathrm{Hom}}_{W'}(C', D) = \underline{\mathrm{Hom}}(C', D)$, and therefore, the Cartesian square obtained by applying the functor $\underline{\mathrm{Hom}}(-, D)$ to the coCartesian square (7.1.3.1), together with the Cartesian square (7.1.1.1), give a Cartesian square of the following form.

$$
\begin{tikzcd}
\underline{\mathrm{Hom}}(C', D) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_W(C, D) \arrow[d] \\
\underline{\mathrm{Hom}}(W', D) \arrow[r] & h(W, D)
\end{tikzcd}
\tag{7.1.3.2}
$$

If we choose a fibrant resolution $W^{-1}C$ of C' in the Joyal model category structure, we get a trivial fibration of the form

$$\underline{\mathrm{Hom}}(W^{-1}C, D) \to \underline{\mathrm{Hom}}(C', D)\,.$$

On the other hand, since the lower horizontal map of diagram (7.1.3.2) is a trivial fibration, so is the upper one, hence the inclusion $\gamma : C \to W^{-1}C$ is a localisation of C by W. By definition, the map γ exhibits $W^{-1}C$ as a representation of the functor $\pi_0(k(\underline{\mathrm{Hom}}_W(C, -)))$ in the homotopy category of the Joyal model category structure. The Yoneda Lemma thus implies that the pair $(W^{-1}C, \gamma)$ is unique up to a unique isomorphism in the homotopy category of the Joyal model category structure. □

• Page 343, there are typos in the definition of a categorical realisation of a décalage: in conditions (iii) and (iv), the functor i should be applied to the diagrams (in examples, the functor i is an inclusion functor, so that this mistake does not affect applications in the book). We should read:

(iii) For any map $f : a \to b$ in A, the commutative square

$$
\begin{tikzcd}
i(a) \arrow[r, "i(\eta_a)"] \arrow[d, "i(f)"'] & i(D(a)) \arrow[d, "i(D(f))"] \\
i(b) \arrow[r, "i(\eta_b)"] & i(D(b))
\end{tikzcd}
\tag{7.3.7.2}
$$

is Cartesian.

(iv) For any object a of A, the commutative square

$$
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & i(\omega) \arrow[d, "i(\pi_a)"] \\
i(a) \arrow[r, "i(\eta_a)"] & i(D(a))
\end{tikzcd}
\tag{7.3.7.3}
$$

is Cartesian.

• Page 357, in Definition 7.4.6, there is an oversight: the map $f : x \to y$ must be in F.