

ÉTALE TAME VANISHING CYCLES OVER $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$

DENIS-CHARLES CISINSKI, MASSIMO PIPPI

ABSTRACT. We develop a theory of tame vanishing cycles for schemes over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ in the context of étale sheaves. We show some desired properties of this formalism, among which: a compatibility with tame vanishing cycles over a (strictly) henselian trait, a compatibility with the theory of tame vanishing cycles over \mathbb{A}_S^1 , a compatibility with tensor product and with duality. In the last section, we prove that monodromy-invariant vanishing cycles, introduced by the second named author, are the homotopy fixed points with respect to a canonical continuous action of μ_∞ of tame vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$.

CONTENTS

Introduction	1
Acknowledgements	4
1. Étale sheaves on Artin stacks	5
2. Recollements	8
3. Review of the theory of tame vanishing cycles	14
4. Tame nearby and vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$	23
5. Functorial behaviour of tame nearby and vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$	29
6. Comparison with vanishing cycles over a strictly henselian trait	36
7. Tame nearby cycles over \mathbb{A}_S^1 and comparison with the étale version of Ayoub's tame nearby cycles	37
8. Compatibility with tensor product and duality	39
9. Tame vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ and monodromy invariant vanishing cycles	42
Appendix A.	47
References	50

INTRODUCTION

The theory of vanishing cycles for the germ of an holomorphic function $(\mathbb{C}^n, \underline{0}) \rightarrow (\mathbb{C}, 0)$ was introduced by J. Milnor in his celebrated book [22]. The theory was developed in the algebraic setup by A. Grothendieck ([10]) and by P. Deligne ([11]). In the latter context, the role of the germ of an holomorphic function is played by a scheme over an henselian trait. The henselian trait here plays the role of a small disk centered at the origin. A variant of the theory of (tame) nearby cycles, developed by J. Ayoub in his monography [3, 4] in the motivic context (more generally, in the context of homotopy stable 2-functors), replaces the small little disk centered at the origin with \mathbb{A}_S^1 . Moreover, Ayoub shows that these two formalisms are compatible in a suitable sense in [5]. Since \mathbb{A}_S^1 is naturally equipped with

an action of the group scheme $\mathbb{G}_{m,S}$, it is natural to seek for a theory of (tame) nearby/vanishing cycles for $\mathbb{G}_{m,S}$ -equivariant functions $X \rightarrow \mathbb{A}_S^1$. This is precisely what we aim to do in this note. We achieve this by using the language of Artin stacks: the datum of a $\mathbb{G}_{m,S}$ -equivariant function $f : X \rightarrow \mathbb{A}_S^1$ is equivalent to that of a morphism $[X/\mathbb{G}_{m,S}] \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ of stacks. In this setup, we let $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ play the role of a little disk centered at the origin. Then, the role of the origin of this little disk is played by the closed substack $B\mathbb{G}_{m,S}$ of $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, while that of the punctured disk by $S \simeq [\mathbb{A}_S^1 - \{0\}/\mathbb{G}_{m,S}]$. The following situation, which was the one of interest in [25], might help to clarify the geometric intuition underlying our theory. Let X be an S -scheme. To provide a morphism $X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ is equivalent to provide a pair (\mathcal{L}, s) , where \mathcal{L} is a line bundle over X and s is a global section of \mathcal{L} . Then, the fiber of X over $B\mathbb{G}_{m,S}$ is the zero locus of s (i.e. the closed subscheme defined by the ideal sheaf generated by the morphism $\mathcal{L}^\vee \rightarrow \mathcal{O}_X$ induced by s), while its fiber over S is the open complementary to this closed subscheme. In the theory developed in [10, 11], a crucial role is played by (the choice of) a separable closure of the fraction field of the base henselian ring. In the variant where the base is \mathbb{A}_S^1 , this role is played by the pro-étale cover $\widetilde{\mathbb{G}_{m,S}} \rightarrow \mathbb{G}_{m,S}$, where $\widetilde{\mathbb{G}_{m,S}} = \varprojlim_{n \in \mathbb{N}_S} (\mathbb{G}_{m,S} \xrightarrow{t \mapsto t^n} \mathbb{G}_{m,S})$, where t is the parameter of $\mathbb{G}_{m,S}$. These are the geometric analogues of an universal cover of the punctured disk in the topological setting. Since in our context the punctured disk is "topologically trivial", this universal cover has to be introduced in an alternative way. The reader might benefit by considering the following topological analogue of our setup: let X be a complex variety endowed with an action of \mathbb{C}^* and let $f : X \rightarrow \mathbb{C}$ be a \mathbb{C}^* -equivariant function. We can consider the following diagram, where all squares are Cartesian:

$$(0.0.1) \quad \begin{array}{ccccccc} f^{-1}(0) & \xrightarrow{i} & X & \xleftarrow{j} & X^* & \xleftarrow{\quad} & \widetilde{X}^* \\ \downarrow & & \downarrow f & & \downarrow f|_{X^*} & & \downarrow \\ \{0\} & \xrightarrow{i} & \mathbb{C} & \xleftarrow{j} & \mathbb{C}^* & \xleftarrow{\text{exp}} & \mathbb{C} \end{array}$$

The two squares on the left form a \mathbb{C}^* -equivariant diagram in a natural way. The exponential map $\text{exp} : \mathbb{C} \rightarrow \mathbb{C}^*$ is a morphism of groups and we let it play the role of the universal cover of the punctured disk. Notice that we can regard the whole diagram as being \mathbb{C} -equivariant thanks to the exponential map. This is also how the *monodromy action* arises in this topological setup.

Remark 0.1. The precise definition of \mathbb{C}^* -equivariant vanishing cycles and its relationship with the formalism introduced here will appear elsewhere.

Clearly, in the algebraic setup we do not dispose of the exponential map. Therefore, we approximate it with the tower of morphisms $\mathbb{G}_{m,S} \xrightarrow{t \mapsto t^n} \mathbb{G}_{m,S}$. In other words, if $f : X \rightarrow \mathbb{A}_S^1$ is an equivariant map of $\mathbb{G}_{m,S}$ -schemes, we look at the tower of equivariant diagrams

$$(0.1.1) \quad \begin{array}{ccccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{j} & U_X & \xleftarrow{\quad} & U_X^{(n)} \\ \downarrow & & \downarrow f & & \downarrow f|_{U_X} & & \downarrow \\ S & \xrightarrow{i} & \mathbb{A}_S^1 & \xleftarrow{j} & \mathbb{G}_{m,S} & \xleftarrow{(-)^n} & \mathbb{G}_{m,S} \end{array}$$

However, $\mathbb{G}_{m,S}$ acts via $\mathbb{G}_{m,S} \xrightarrow{(-)^n} \mathbb{G}_{m,S}$ on the two left squares of this diagram. Therefore, taking the limit over all n 's, we obtain a $\widetilde{\mathbb{G}_{m,S}}$ -equivariant diagram. This is how we obtain the *monodromy action* in this setting.

In the language of stacks, this is obtained by looking at the following diagrams

$$(0.1.2) \quad \begin{array}{ccccc} \mathrm{B}\mathbb{G}_{m,S} & \xrightarrow{i} & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \xleftarrow{j} & S \\ \Theta_0^{(n)} \downarrow & & \downarrow \Theta^{(n)} & & \downarrow id_S \\ \mathrm{B}\mathbb{G}_{m,S} & \xrightarrow{i} & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \xleftarrow{j} & S, \end{array}$$

where $\Theta^{(n)}$ (resp. $\Theta_0^{(n)}$), at the level of T -points, induces the functor $(\mathcal{L}, s) \mapsto (\mathcal{L}^{\otimes n}, s^{\otimes n})$ (resp. $\mathcal{L} \mapsto \mathcal{L}^{\otimes n}$).

For an Artin stack X endowed with a morphism $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, we introduce three ∞ -functors

$$(0.1.3) \quad \Psi_{p,\eta}^t : \mathcal{D}_{\text{ét}}(U_X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty},$$

$$(0.1.4) \quad \Psi_p^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{V}_p; \Lambda),$$

$$(0.1.5) \quad \Phi_p^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty},$$

that we call respectively: *tame nearby cycles*, *total tame nearby cycles* and *tame vanishing cycles*. Here $\mathcal{D}_{\text{ét}}(X; \Lambda)$ (resp. $\mathcal{D}_{\text{ét}}(X_0; \Lambda)$) denotes the stable ∞ -category of complexes of étale sheaves on X (resp. X_0). The objects of the stable ∞ -category $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ are, roughly, complexes of étale sheaves on X_0 endowed with a *continuous* action of $\mu_\infty = \varprojlim \mu_n$. Finally, $\mathcal{D}_{\text{ét}}(\mathcal{V}_p; \Lambda)$ is a stable ∞ -category whose objects are, roughly, μ_∞ -equivariant morphisms $F \rightarrow G$, where F is endowed with the trivial action.

Applications and related works. Our formalism is intimately related to that of *monodromy-invariant vanishing cycles*, introduced by the second named author in [24, 25]. The precise relationship between these two constructions is spelled out in §9. In a forthcoming paper, we will introduce the formalism of tame nearby/vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ in the context of étale motives (as considered by Ayoub [5], or equivalently, of h-motives, as developed by the first named author in collaboration with F. Déglise [8]). In fact, the present paper can be interpreted as such a formalism in the case where the algebra of coefficients is torsion (the case of rational coefficients is much more technical: not only because it must take place in the context of motivic sheaves, but also because there are obstructions we must deal with that vanish with torsion coefficients). The idea of considering $\mathbb{G}_{m,S}$ -equivariant nearby cycles goes back implicitly at least to Verdier [28], in his work on monodromic sheaves that appear through his specialization functor, defined via deformation to the normal cone, and the present article is part of an ongoing series of papers devoted to developing microlocal methods for ℓ -adic sheaves as well as for motivic sheaves.

Outline of the paper. The paper is organised as follows:

- In section 1 we quickly recollect the formalism of Grothendieck's 6 functors formalism for stable ∞ -categories of étale complexes on (derived) Artin stacks.

- In section 2, we quickly review the theory of recollements in the context of ∞ -categories as developed by J. Lurie in [21] and we provide certain constructions and results that we were not able to locate in the existing literature. In particular, we provide a method to construct an adjunction between two recollements, starting from the datum of two adjunctions between the subcategories that define them.
- In section 3 we reinterpret the classical theory of tame vanishing cycles over a strictly henselian trait in the language of (pro) Deligne-Mumford stacks. We decided to include this section to emphasize that the language of stacks is well suited to develop a theory of vanishing cycles.
- Section 4 is the core of this paper. Here we construct the three ∞ -functors $\Psi_{p,\eta}^t$, Ψ_p^t and Φ_p^t that we mentioned in the introduction. Moreover, in section 5 we exhibit certain natural compatibility properties with the $*$ -pullback, $*$ -pushforward, $!$ -pullback and $!$ -pushforward.
- Relying on the re-interpretation of the theory of tame nearby cycles over a strictly henselian trait provided in section 3, in section 6 we show how our formalism of tame nearby vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ generalises the classical theory.
- In section 7, we show that if the morphism $X \rightarrow \mathbb{A}_S^1$ factors through $\mathbb{A}_S^1 \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, we re-find the étale version of the formalism of tame nearby cycles over \mathbb{A}_S^1 developed by J. Ayoub by composing our functor $\Psi_{p,\eta}^t$ with the forgetful functor $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$.
- Building on the results of section 7 and on the results provided by J. Ayoub for his formalism, in section 8 we show that $\Psi_{p,\eta}^t$ is compatible with tensor products and with duality.
- In section 9, we prove that monodromy invariant vanishing cycles introduced by the first named author are nothing but the homotopy fixed points of the μ_∞ -action on the sheaf of tame vanishing cycles introduced in section 4. In particular, we show how the first Chern class of the line bundle defining the morphism $X_0 \rightarrow \text{B}\mathbb{G}_{m,S}$ plays an important role in the canonical action of μ_∞ on the sheaf of tame vanishing cycles.

Notation.

- \mathcal{S} denotes the ∞ -category of spaces;
- for a simplicial set X , we denote by $X([n])$ the set of n -simplices of X ;
- whenever we will have a diagram $Y \xrightarrow{f} X \xleftarrow{g} Z$ in a (∞ -)category with finite limits, we will denote the fiber product by $Y \times_{f,X,g} Z$ if we want to keep track of f and g in the notation;
- we will denote the ∞ -category of stable and presentable ∞ -categories with right adjoints as $\mathbf{Pr}_{\text{stb}}^{\text{R}}$. Similarly, we denote the ∞ -category of stable and presentable ∞ -categories with left adjoints as $\mathbf{Pr}_{\text{stb}}^{\text{L}}$;
- unless otherwise specified, by 2-category we will always mean "strict (2, 1)-category";
- all functors will be implicitly derived;
- following [20], we denote the simplicial nerve by N .

ACKNOWLEDGEMENTS

MP wishes to thank Bertrand Toën and Gabriele Vezzosi for many conversations on the subject of this paper.

This project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation programme (grant agreement No. 725010).

The authors are supported by the collaborative research center SFB 1085 *Higher Invariants - Interactions between Arithmetic Geometry and Global Analysis* funded by the Deutsche Forschungsgemeinschaft.

1. ÉTALE SHEAVES ON ARTIN STACKS

One of the major achievements of [14, 13, 2] is the construction of *Grothendieck's six functors formalism* on the derived categories of étale sheaves of schemes. In this paper, it is crucial that this formalism can be extended to Artin stacks. Many papers have been dedicated to this matter, including [18, 6, 23, 17, 19]. We will briefly recall what is the derived category of an Artin stack and how the assignment $X \mapsto \mathcal{D}_{\text{ét}}(X, \Lambda)$ is endowed with a six functor formalism following [19]. Here Λ denotes a fixed ring of coefficients, that we suppose to be torsion and such that for all points s of the base scheme S , $\text{char}(k(s))$ is invertible in Λ . Notice that, in particular, we will use the language of ∞ -categories: $\mathcal{D}_{\text{ét}}(X; \Lambda)$ is a presentable, stable ∞ -category. Recall that, for an Artin stack X and a smooth atlas $X' \rightarrow X$ with X' a scheme, $\mathcal{D}_{\text{ét}}(X; \Lambda)$ is equivalent to the limit of the following diagram

$$(1.0.1) \quad \mathcal{D}_{\text{ét}}(X'; \Lambda) \rightrightarrows \mathcal{D}_{\text{ét}}(X' \times_X X'; \Lambda) \begin{array}{c} \rightrightarrows \\ \rightrightarrows \end{array} \mathcal{D}_{\text{ét}}(X' \times_X X' \times_X X'; \Lambda) \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} \cdots$$

This construction does not depend on the atlas $X' \rightarrow X$.

Grothendieck's six functors formalism consists of the following data and formulas:

- For every morphisms of Artin stacks $f : Y \rightarrow X$, there is an adjunction

$$(1.0.2) \quad f^* : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(Y; \Lambda) : f_*$$

where the left adjoint f^* is called *pullback* (or **-pullback*) and the right adjoint f_* is called *pushforward* (or **-pushforward*).

- For every separated morphism of finite type, there is an adjunction

$$(1.0.3) \quad f_! : \mathcal{D}_{\text{ét}}(Y; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(X; \Lambda) : f^!$$

where the left adjoint $f_!$ is called *exceptional pushforward* (or *!-pushforward*) and the right adjoint $f^!$ is called *exceptional pullback* (or *!-pullback*).

- For every X , $\mathcal{D}_{\text{ét}}(X; \Lambda)$ is a closed symmetric monoidal ∞ -category. The tensor product is denoted by $- \otimes_X -$ and the internal hom by $\underline{\text{Hom}}_X$.
- The assignments $f \mapsto f^*$, $f \mapsto f_*$, $f \mapsto f_!$ and $f \mapsto f^!$ are functorial (in the ∞ -categorical sense) in f .
- There is a natural transformation of ∞ -functors

$$(1.0.4) \quad f_! \rightarrow f_*$$

that is an equivalence whenever $f : X \rightarrow Y$ is representable by Deligne-Mumford stacks, there exists a finite surjection $Z \rightarrow X$ with Z an algebraic space and f is proper.

- For every smooth separated morphism $f : X \rightarrow Y$ of relative dimension d , there is a natural equivalence of ∞ -functors

$$(1.0.5) \quad f^* \simeq f^!(-d)[-2d],$$

where $(-d)$ denotes the *Tate shift* while $[-2d]$ the *cohomological shift*.

- **(Künneth Formula)** For every finite collection of morphisms $\{f_i : X_i \rightarrow Y_i\}_{\{i=1,\dots,n\}}$ of Artin stacks, given a Cartesian square

$$(1.0.6) \quad \begin{array}{ccc} X & \xrightarrow{(p_1, \dots, p_n)} & X_1 \times \dots \times X_n \\ \downarrow f & & \downarrow f_1 \times \dots \times f_n \\ Y & \xrightarrow{(q_1, \dots, q_n)} & Y_1 \times \dots \times Y_n \end{array}$$

the following square commutes (in the ∞ -categorical sense)

$$(1.0.7) \quad \begin{array}{ccc} \mathcal{D}_{\text{ét}}(X_1; \Lambda) \times \dots \times \mathcal{D}_{\text{ét}}(X_n; \Lambda) & \xrightarrow{p_1^* \otimes \dots \otimes p_n^*} & \mathcal{D}_{\text{ét}}(X; \Lambda) \\ \downarrow f_1! \times \dots \times f_n! & & \downarrow f! \\ \mathcal{D}_{\text{ét}}(Y_1; \Lambda) \times \dots \times \mathcal{D}_{\text{ét}}(Y_n; \Lambda) & \xrightarrow{q_1^* \otimes \dots \otimes q_n^*} & \mathcal{D}_{\text{ét}}(Y; \Lambda). \end{array}$$

There are two particularly important cases that deserve a name on their own:

- (1) **(Base Change Formulas)** If $n = 1$, then we are just considering a Cartesian square

$$(1.0.8) \quad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow g' & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

and the Künneth Formula implies that we have an equivalence of ∞ -functors

$$(1.0.9) \quad f'_! \circ g'^* \simeq g^* \circ f_!.$$

Notice that, by the uniqueness (up to a contractible space of choices) of right adjoints, this formally implies

$$(1.0.10) \quad g'_* \circ f'^! \simeq f^! \circ g_*.$$

In particular, if f is proper, representable by Deligne-Mumford stacks and X admits a finite surjection from an algebraic space, we obtain the **proper base change formula**

$$(1.0.11) \quad f'_* \circ g'^* \simeq g^* \circ f_*,$$

while if f is smooth we obtain the **smooth base change formula**

$$(1.0.12) \quad g'_* \circ f'^* \simeq f^* \circ g_*.$$

- (2) **(Projection Formula)** For every morphism $f : X \rightarrow Y$, we can consider the pullback square

$$(1.0.13) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow (id, f) & & \downarrow \Delta_Y \\ X \times Y & \xrightarrow{f \times id} & Y \times Y, \end{array}$$

in which case the Künneth formula translates as the equivalence

$$(1.0.14) \quad f_!(- \otimes f^*(-)) \simeq f_!(-) \otimes -.$$

Notice that the Projection Formula immediately implies the following:

$$(1.0.15) \quad \underline{\mathrm{Hom}}_Y(f_!(-), -) \simeq f_* \underline{\mathrm{Hom}}_X(-, f^!(-)),$$

$$(1.0.16) \quad f^! \underline{\mathrm{Hom}}_Y(-, -) \simeq \underline{\mathrm{Hom}}_X(f^*(-), f^!(-)).$$

- **(Localization property)** If $i : Z \rightarrow X$ is a closed embedding and $j : U \rightarrow X$ is its open complementary, there is a fiber-cofiber sequence of ∞ -functors

$$(1.0.17) \quad j_! \circ j^! \rightarrow id \rightarrow i_* \circ i^*.$$

It is known (see [9, Proposition 2.3.3]) that this implies that

$$(1.0.18) \quad i_* \circ i^! \rightarrow id \rightarrow j_* \circ j^*$$

is a fiber-cofiber sequence of ∞ -functors as well.

- **(Absolute Purity)** If $i : Z \rightarrow X$ is a closed embedding of regular stacks of constant codimension c , there is an equivalence

$$(1.0.19) \quad \underline{\Lambda}_Z(-c)[-2c] \simeq i^! \underline{\Lambda}_X.$$

- **(Duality)** Let $f : X \rightarrow S$ be a morphism from an Artin stack to a regular scheme S . Put $K_X := f^!(\underline{\Lambda}_S)$ and consider the ∞ -functor

$$(1.0.20) \quad \mathbb{D}_X := \underline{\mathrm{Hom}}_X(-, K_X) : \mathcal{D}_{\text{ét}}(X; \Lambda)^{\text{op}} \rightarrow \mathcal{D}_{\text{ét}}(X; \Lambda).$$

Then K_X is a *dualizing object* in $\mathcal{D}_{\text{ét}}(X; \Lambda)$. That is, for every $M \in \mathcal{D}_{\text{ét}}(X; \Lambda)$, the canonical morphism

$$(1.0.21) \quad M \rightarrow \mathbb{D}_X(\mathbb{D}_X(M))$$

is an equivalence. Moreover, the *duality functor* \mathbb{D}_X satisfies the following compatibilities with the six functors. Let $f : X \rightarrow Y$ and let $L, M \in \mathcal{D}_{\text{ét}}(X; \Lambda)$, $N \in \mathcal{D}_{\text{ét}}(Y; \Lambda)$:

$$(1.0.22) \quad \mathbb{D}_X(L \otimes_X \mathbb{D}_X(M)) \simeq \underline{\mathrm{Hom}}_X(L, M);$$

$$(1.0.23) \quad \mathbb{D}_X(f^*(N)) \simeq f^! \mathbb{D}_Y(N);$$

$$(1.0.24) \quad f^* \mathbb{D}_Y(N) \simeq \mathbb{D}_X(f^! N);$$

$$(1.0.25) \quad \mathbb{D}_Y(f_* M) \simeq f_! \mathbb{D}_X(M);$$

$$(1.0.26) \quad f_! \mathbb{D}_X(M) \simeq \mathbb{D}_Y(f_* M).$$

The six functor formalism is expressed by the existence of a lax monoidal ∞ -functor

$$(1.0.27) \quad \mathrm{Corr}(N(\mathbf{Art}_S))_{F, \text{all}}^{\otimes} \rightarrow \mathbf{Pr}_{\text{stb}}^{\text{L}, \otimes},$$

which encodes (in an homotopy coherent way!) all the properties listed above. Here F denotes the class of separated morphisms locally of finite type, while $\mathrm{Corr}(\mathbf{Art}_S)_{F, \text{all}}^{\otimes}$ is the symmetric monoidal

∞ -category denoted by $N(\mathbf{Art}_S)_{\text{corr}:\mathcal{E}_F,\text{all}}^{\otimes}$ in [19]. Roughly, this ∞ -functor sends an Artin stack X to $\mathcal{D}_{\text{ét}}(X; \Lambda)$ and a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ to

$$(1.0.28) \quad f_! \circ g^* : \mathcal{D}_{\text{ét}}(Y; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X; \Lambda).$$

We refer to *loc. cit.* for further details.

Also notice that passing to right adjoints, i.e. through the equivalence $\mathbf{Pr}_{\text{stb}}^{\text{L},\otimes} \simeq (\mathbf{Pr}_{\text{stb}}^{\text{R},\otimes})^{\text{op}}$, we get an ∞ -functor

$$(1.0.29) \quad \text{Corr}(N(\mathbf{Art}_S))_{\text{all},F}^{\otimes} \rightarrow \mathbf{Pr}_{\text{stb}}^{\text{R},\otimes}$$

which sends a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ to

$$(1.0.30) \quad g_* \circ f^! : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y; \Lambda).$$

2. RECOLLEMENTS

In this section we will quickly review the theory of *recollement* of ∞ -categories. We will follow [21, Appendix A.8]. To the authors knowledge, the notion of recollement first appeared in [14, 13, 2] in the context of topoi. All results and constructions of this section are well known (even those that might not appear in the literature). The example which motivates the theory of recollement is the following one: let X be a topological space. Let $j : U \subseteq X$ be an open subset and $i : Z = X - U \subseteq X$ be its closed complementary. Let $\mathcal{F} \in \mathbf{Shv}(X)$ be a sheaf (of sets, say) on X . It determines the following triplet: $(\mathcal{F}|_Z := i^*\mathcal{F}, \mathcal{F}|_U := j^*\mathcal{F}, \mathcal{F}|_Z \rightarrow i^*j_*\mathcal{F}|_U)$. Such triplets can be organized in a category in an evident way, denoted $(\mathbf{Shv}(Z), \mathbf{Shv}(U), i^*j_* : \mathbf{Shv}(U) \rightarrow \mathbf{Shv}(Z))$ and the assignment

$$\mathcal{F} \mapsto (\mathcal{F}|_Z, \mathcal{F}|_U, \mathcal{F}|_Z \rightarrow i^*j_*\mathcal{F}|_U)$$

determines a functor

$$\mathbf{Shv}(X) \rightarrow (\mathbf{Shv}(Z), \mathbf{Shv}(U), i^*j_* : \mathbf{Shv}(U) \rightarrow \mathbf{Shv}(Z)).$$

Moreover, this functor happens to be an equivalence. For example, we can recover the sheaf \mathcal{F} (up to isomorphism) from $(\mathcal{F}|_Z, \mathcal{F}|_U, \mathcal{F}|_Z \rightarrow i^*j_*\mathcal{F}|_U)$ by taking the limit of the diagram

$$\begin{array}{c} j_*\mathcal{F}|_U \\ \downarrow \\ i_*\mathcal{F}|_Z \rightarrow i_*i^*j_*\mathcal{F}|_U. \end{array}$$

The key properties of the phenomenon described above are highlighted in the following

Definition 2.1. [21, Definition A.8.1] Let \mathcal{C} be an ∞ -category with finite limits and let $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ be two full subcategories. We say that \mathcal{C} is the *recollement* of \mathcal{C}_0 and \mathcal{C}_1 if the following hypothesis are met:

- (1) \mathcal{C}_0 and \mathcal{C}_1 are stable under equivalences;
- (2) the inclusions $i_0 : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $i_1 : \mathcal{C}_1 \hookrightarrow \mathcal{C}$ admit left adjoints $L_0 : \mathcal{C} \rightarrow \mathcal{C}_0$ and $L_1 : \mathcal{C} \rightarrow \mathcal{C}_1$;
- (3) L_0 and L_1 are left exact;
- (4) $L_1 \circ i_0$ is equivalent to the constant functor $\mathcal{C}_0 \rightarrow \mathcal{C}_1$ determined by a final object of \mathcal{C}_1 (which exists as \mathcal{C} has finite limits and L_1 is left exact);

- (5) (L_0, L_1) detects equivalences, i.e. an arrow $\alpha : x \rightarrow y$ in \mathcal{C} is an equivalence if and only if $L_0(\alpha)$ and $L_1(\alpha)$ are.

Example 2.2. Let X, U, Z, i, j be as above. Let $\mathbf{Shv}(X)$ (resp. $\mathbf{Shv}(U)$, resp. $\mathbf{Shv}(Z)$) be the ∞ -category of \mathcal{S} -valued sheaves on X (resp. U , resp. Z). The functors i_* and j_* are fully faithful and the full subcategories stable under equivalences determined by these functors are such that $\mathbf{Shv}(X)$ is a recollement of those.

Remark 2.3. It is proved in [21, Appendix A.8] that the theory of recollements is equivalent to that of left exact fibrations (see [21, A.8.12]). Recall that a *left exact fibration* is a morphism of simplicial sets $p : \mathcal{M} \rightarrow \Delta^1$ such that

- (1) p is a Cartesian fibration;
- (2) \mathcal{M}_0 and \mathcal{M}_1 admit finite limits;
- (3) the ∞ -functor $\mathcal{M}_1 \rightarrow \mathcal{M}_0$ determined by p is left exact.

Notation 2.4. According to the equivalence between the theory of recollements and that of left exact fibrations, it is clear that an adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$ between ∞ -categories with finite limits determines an ∞ -category that is a recollement of \mathcal{C} and \mathcal{D} . We will refer to such ∞ -category as the *recollement determined by the adjunction* $\mathcal{C} \rightleftarrows \mathcal{D}$.

Example 2.5. Notice that every ∞ -category with finite limits \mathcal{C} is the recollement of \mathcal{C} and Δ^0 . Let $p : \mathcal{C}^\triangleright \rightarrow \Delta^1 = (\mathcal{C} \rightarrow \Delta^0) \star \Delta^0$ (see [20] for notation). Then p is a left exact fibration:

- (1) $p^{-1}(0) = \mathcal{C}$ has finite limits by assumption. Clearly, $p^{-1}(1) = \Delta^0$ has finite limits;
- (2) to show that p is a Cartesian fibration it suffices to show that there exists a Cartesian edge over $0 \rightarrow 1$. Let v denote the (unique) object of $\mathcal{C}^\triangleright$ over $1 \in \Delta^1$ and let f be a final object of \mathcal{C} . Then $f \rightarrow v$ is a Cartesian edge over $0 \rightarrow 1$;
- (3) the ∞ -functor $\Delta^0 \rightarrow \mathcal{C}$ determined by p is equivalent to $f : \Delta^0 \rightarrow \mathcal{C}$. In particular, it is left exact.

By the results in [21, Appendix A.8] we see that $\mathcal{C} \simeq \text{Fun}_{\Delta^1}(\Delta^1, \mathcal{C}^\triangleright)$ is the recollement of $\mathcal{C} \simeq \{s \in \text{Fun}_{\Delta^1}(\Delta^1, \mathcal{C}^\triangleright) : s(1) \text{ is a final object of } p^{-1}(1)\}$ and $\Delta^0 \simeq \{s \in \text{Fun}_{\Delta^1}(\Delta^1, \mathcal{C}^\triangleright) : s \text{ is a Cartesian edge}\}$.

We will be interested in the following ∞ -categorical analogue of [14, Exercise 9.5.11]:

Proposition 2.6. *Let \mathcal{C} and \mathcal{D} be ∞ -categories. Assume that \mathcal{D} is the recollement of $\mathcal{D}_0, \mathcal{D}_1 \subseteq \mathcal{D}$. Then $\text{Fun}(\mathcal{C}, \mathcal{D})$ is the recollement of $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ and $\text{Fun}(\mathcal{C}, \mathcal{D}_1)$.*

Proof. $\text{Fun}(\mathcal{C}, \mathcal{D})$ admits finite limits by (the dual of) [20, Corollary 5.1.2.3 (1)]. $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ is a full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ as it is a limit of the following diagram

$$\begin{array}{ccc} & \text{Fun}(\mathcal{C}, \mathcal{D}) & \\ & \downarrow & \\ \Pi_{\mathcal{C}([0])} \mathcal{D}_0 & \hookrightarrow & \Pi_{\mathcal{C}([0])} \mathcal{D}, \end{array}$$

where the vertical arrow is induced by evaluations at all objects $c \in \mathcal{C}([0])$ and the bottom arrow is induced by $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ and is fully faithful. An analogous argument shows that $\text{Fun}(\mathcal{C}, \mathcal{D}_1)$ is

a full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ as well. The fact that $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ and $\text{Fun}(\mathcal{C}, \mathcal{D}_1)$ are stable under equivalences is straightforward as $\mathcal{D}_0 \subseteq \mathcal{D}$ and $\mathcal{D}_1 \subseteq \mathcal{D}$ have this property. The inclusions $\text{Fun}(\mathcal{C}, \mathcal{D}_0), \text{Fun}(\mathcal{C}, \mathcal{D}_1) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ are induced by $i_0 : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ and by $i_1 : \mathcal{D}_1 \hookrightarrow \mathcal{D}$ and admit $L_0 \circ - : \text{Fun}(\mathcal{C}, \mathcal{D}_0) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$ and $L_1 \circ - : \text{Fun}(\mathcal{C}, \mathcal{D}_1) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$ as left adjoints. These are left exact as by (the dual of) [20, Corollary 5.1.2.3 (2)], limits in $\text{Fun}(\mathcal{C}, \mathcal{D})$ are computed pointwise. This also implies that the functor $\text{Fun}(\mathcal{C}, \mathcal{D}_0) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}_1)$ is equivalent to the constant functor determined by a final object of $\text{Fun}(\mathcal{C}, \mathcal{D}_1)$. Finally, $(L_0 \circ -, L_1 \circ -)$ detects equivalences as a morphism in $\text{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence if and only if it is pointwise true. \square

Corollary 2.7. *Let \mathcal{C}, \mathcal{D} be ∞ -categories. Assume that \mathcal{D} is the recollement of \mathcal{D}_0 and \mathcal{D}_1 . The datum of functors $f_0 : \mathcal{C} \rightarrow \mathcal{D}_0$, $f_1 : \mathcal{C} \rightarrow \mathcal{D}_1$ and of a morphism $f_0 \rightarrow L_0 \circ i_1 \circ f_1$ in $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ determines a functor $f \in \text{Fun}(\mathcal{C}, \mathcal{D})$ such that $f_0 \simeq L_0 \circ f$ and $f_1 \simeq L_1 \circ f$.*

Proof. By the previous proposition, $\text{Fun}(\mathcal{C}, \mathcal{D})$ is the recollement of $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ and $\text{Fun}(\mathcal{C}, \mathcal{D}_1)$. Let $\mathcal{M}_0 = \text{Fun}(\mathcal{C}, \mathcal{D}_0)$, \mathcal{M}_1 be the full subcategory of $\text{Fun}(\Delta^1, \text{Fun}(\mathcal{C}, \mathcal{D}))$ spanned by morphisms $\alpha : g \rightarrow g'$ such that $g \in \text{Fun}(\mathcal{C}, \mathcal{D}_1)$ and α exhibits g' as a $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ -localization of g . The functor "evaluation at 1" $\mathcal{M}_1^{\text{op}} \rightarrow \mathcal{M}_0^{\text{op}}$ determines a morphism $\Theta : [1] \rightarrow \text{Set}_\Delta$. Let $\mathcal{M} := \text{N}_\Theta([1])^{\text{op}}$ (see [20, Definition 3.2.5.2]). By [21, Proposition A.8.11], there is a left exact fibration $\mathcal{M} \rightarrow \Delta^1$ and is the one associated to the recollement of $\text{Fun}(\mathcal{C}, \mathcal{D}_0)$ and $\text{Fun}(\mathcal{C}, \mathcal{D}_1)$. Unravelling the definitions, $(f_0, f_1, f_0 \rightarrow L_0 \circ i_1 \circ f_1)$ is the datum of an object in $\text{Fun}_{\Delta^1}(\Delta^1, \mathcal{M})$, which by [21, Proposition A.8.8] corresponds to a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ ($\Delta^1 \xrightarrow{id} \Delta^1$ is the left exact fibration associated to the recollement of Δ^0 and Δ^0 , which is equivalent to Δ^0). The last assertion is clear. \square

Moreover, we will be interested in the following situation.

Construction 2.8. Let \mathcal{D} be the recollement of \mathcal{D}_0 and \mathcal{D}_1 . As usual, we will denote by i_0 (resp. i_1) the fully faithful embedding $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ (resp. $\mathcal{D}_1 \hookrightarrow \mathcal{D}$) and by L_0 (resp. L_1) its left adjoint. We will further assume that the ∞ -functor $L_0 \circ i_1 : \mathcal{D}_1 \rightarrow \mathcal{D}_0$ admits a left adjoint $U : \mathcal{D}_0 \rightarrow \mathcal{D}_1$.

Notice that the adjunction $id : \mathcal{D}_1 \rightleftarrows \mathcal{D}_1 : id$ induces a left exact fibration $\mathcal{M} \rightarrow \Delta^1$. The associated ∞ -category is $\text{Fun}(\Delta^1, \mathcal{D}_1)$, recollement of \mathcal{D}_1 and \mathcal{D}_1 . The relevant ∞ -functors are

$$\mathcal{D}_1 \begin{array}{c} \xrightarrow{j_0} \\ \xleftarrow{R_0} \end{array} \text{Fun}(\Delta^1, \mathcal{D}_1) \begin{array}{c} \xrightarrow{R_1} \\ \xleftarrow{j_1} \end{array} \mathcal{D}_1,$$

defined at the level of objects as follows (e denotes a final object of \mathcal{D}_1):

$$\begin{aligned} j_0(d) &= (d \rightarrow e), & R_0(d_0 \rightarrow d_1) &= d_0, \\ j_1(d) &= (d \xrightarrow{id} d), & R_1(d_0 \rightarrow d_1) &= d_1. \end{aligned}$$

Notice that $R_1 \circ j_0(d) = e$ and $R_0 \circ j_1 \simeq id$.

We produce a morphism $sp_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}_1)$ as follows: consider

- $U \circ L_0 : \mathcal{D} \rightarrow \mathcal{D}_1$;
- $L_1 : \mathcal{D} \rightarrow \mathcal{D}_1$;
- the natural transformation $U \circ L_0 \rightarrow L_1$ corresponding, under the adjunction $(U, L_0 \circ i_1)$, to $L_0 \rightarrow L_0 \circ i_1 \circ L_1 \simeq L_0 \circ (id \xrightarrow{unit} i_1 \circ L_1)$.

By Corollary 2.7, these data determine an ∞ -functor

$$sp_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}_1).$$

Remark 2.9. We will be particularly interested in the composition of $sp_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Fun}(\Delta^1, \mathcal{D}_1)$ with the ∞ -functor

$$fiber : \text{Fun}(\Delta^1, \mathcal{D}_1) \rightarrow \mathcal{D}_1,$$

right adjoint to the ∞ -functor

$$j_0 : \mathcal{D}_1 \rightarrow \text{Fun}(\Delta^1, \mathcal{D}_1).$$

Adjunctions between recollements. In this subsection we will study in some detail how to produce adjunctions between recollements.

Notation 2.10. In this section, we will consider two ∞ -categories \mathcal{C} and \mathcal{D} such that \mathcal{C} is the recollement of two subcategories \mathcal{C}_0 and \mathcal{C}_1 and \mathcal{D} is the recollement of two subcategories \mathcal{D}_0 and \mathcal{D}_1 . Moreover, we will label $i_k : \mathcal{C}_k \rightarrow \mathcal{C}$ (resp. $j_k : \mathcal{D}_k \rightarrow \mathcal{D}$) the fully faithful embedding and $L_k : \mathcal{C} \rightarrow \mathcal{C}_k$ (resp. $R_k : \mathcal{D} \rightarrow \mathcal{D}_k$) its left adjoint ($k = 0, 1$).

We will assume that we are given two adjunctions $F_0 : \mathcal{C}_0 \rightleftarrows \mathcal{D}_0 : G_0$ and $F_1 : \mathcal{C}_1 \rightleftarrows \mathcal{D}_1 : G_1$ and homotopies

$$(2.10.1) \quad \begin{array}{ccc} \mathcal{C}_0 & \xleftarrow{L_0 \circ i_1} & \mathcal{C}_1 \\ F_0 \downarrow & & \downarrow F_1 \\ \mathcal{D}_0 & \xleftarrow{R_0 \circ j_1} & \mathcal{D}_1 \end{array} \quad \begin{array}{ccc} \mathcal{C}_0 & \xleftarrow{L_0 \circ i_1} & \mathcal{C}_1 \\ G_0 \uparrow & & \uparrow G_1 \\ \mathcal{D}_0 & \xleftarrow{R_0 \circ j_1} & \mathcal{D}_1. \end{array}$$

Our aim will be to construct an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ out of these data.

We will start with an explicit description of a recollement. Consider the following variant of Construction 2.8:

Construction 2.11. As we have already observed, $\text{Fun}(\Delta^1, \mathcal{C}_0)$ is the recollement of \mathcal{C}_0 and \mathcal{C}_0 (corresponding to the left exact fibration given by the adjunction $id : \mathcal{C}_0 \rightleftarrows \mathcal{C}_0 : id$). Then, by Corollary 2.7, $(L_0 : \mathcal{C} \rightarrow \mathcal{C}_0, L_0 \circ i_1 \circ L_1 : \mathcal{C} \rightarrow \mathcal{C}_0, L_0 \rightarrow L_0 \circ i_1 \circ L_1)$ corresponds to an ∞ -functor

$$(2.11.1) \quad \overline{sp}_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Fun}(\Delta^1, \mathcal{C}_0).$$

Proposition 2.12. *Let $(\{\mathcal{C}\}_{k=0,1}, \{i_k\}_{k=0,1}, \{L_k\}_{k=0,1})$ be as above. The square*

$$(2.12.1) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{L_1} & \mathcal{C}_1 \\ \downarrow \overline{sp}_{\mathcal{C}} & & \downarrow L_0 \circ i_1 \\ \text{Fun}(\Delta^1, \mathcal{C}_0) & \xrightarrow{ev_1} & \mathcal{C}_0, \end{array}$$

where ev_1 denotes the "evaluation at 1" ∞ -functor, is Cartesian.

Proof. We shall prove that $\text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$ is the recollement of \mathcal{C}_0 and \mathcal{C}_1 . First notice that $\text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$ has finite limits.

Let $p : \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \text{Fun}(\Delta^1, \mathcal{C}_0)$, $q : \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$ and $r \simeq ev_1 \circ p \simeq L_0 \circ i_1 \circ q : \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_0$ denote the canonical morphisms. Let

- $l_0 : \mathcal{C}_0 \rightarrow \mathcal{C} \rightarrow \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$;
- $l_1 : \mathcal{C}_1 \rightarrow \mathcal{C} \rightarrow \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$.

Recall that, for every couple of objects x, y in $\text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$, the canonical morphism

$$(2.12.2) \quad \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1}(x, y) \rightarrow \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)) \times_{\text{Map}_{\mathcal{C}_0}(r(x), r(y))} \text{Map}_{\mathcal{C}_1}(q(x), q(y))$$

is a weak homotopy-equivalence. It is then easy to see that l_0 and l_1 are fully faithful. Indeed, if x, y are objects in \mathcal{C}_0 , then $q(x) \simeq q(y) \simeq 0$ (final object in \mathcal{C}_1). In particular,

$$(2.12.3) \quad \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)) \times_{\text{Map}_{\mathcal{C}_0}(r(x), r(y))} \text{Map}_{\mathcal{C}_1}(q(x), q(y)) \simeq \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)).$$

Moreover, for such x, y ,

$$(2.12.4) \quad \text{Map}_{\mathcal{C}}(x, y) \simeq \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)).$$

If $x, y \in \mathcal{C}_1$, it is easy to see that

$$(2.12.5) \quad \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)) \simeq \text{Map}_{\mathcal{C}_0}(r(x), r(y))$$

(as $p(x) = L_0 \circ i_1(x) \xrightarrow{id} L_0 \circ i_1(x)$). Therefore,

$$(2.12.6) \quad \text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)) \times_{\text{Map}_{\mathcal{C}_0}(r(x), r(y))} \text{Map}_{\mathcal{C}_1}(q(x), q(y)) \simeq \text{Map}_{\mathcal{C}_1}(q(x), q(y))$$

is a weak homotopy-equivalence.

Moreover, the functors

$$(2.12.7) \quad ev_0 \circ p : \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_0, \quad q : \text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$$

are left adjoints to l_0 and l_1 respectively. It is also clear that these functors are left exact.

The functor $q \circ l_0$ is equivalent to $L_1 \circ i_0$ and therefore it takes every object of \mathcal{C}_0 to the final object in \mathcal{C}_1 .

We are left to show that $(ev_0 \circ p, q)$ detect equivalences.

An object in $\text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$ is a triplet

$$(2.12.8) \quad x = (x_0 \rightarrow x'_0, x_1, \alpha_x : x'_0 \simeq L_0 \circ i_1(x_1)),$$

where $x_0 \rightarrow x'_0$ is an arrow in \mathcal{C}_0 and x_1 is an object in \mathcal{C}_1 .

A morphism between two objects x, y in $\text{Fun}(\Delta^1, \mathcal{C}_0) \times_{\mathcal{C}_0} \mathcal{C}_1$ is a triplet

$$(2.12.9) \quad \left(\sigma : \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}_0, g, \tau : \sigma_{\{1\} \times \Delta^1} \simeq L_0 \circ i_1(g) \right),$$

where $\sigma_{\Delta^1 \times \{0\}} = (x_0 \rightarrow x'_0)$, $\sigma_{\Delta^1 \times \{1\}} = (y_0 \rightarrow y'_0)$ and $g : x_1 \rightarrow y_1$. Such a morphism is an equivalence if and only if $\sigma_{\{0\} \times \Delta^1}$ and g are equivalences (as $\sigma_{\{1\} \times \Delta^1} \simeq L_0 \circ i_1(g)$). Using this description and the fact that $ev_0 \circ p$ and q send such triplet to $\sigma_{\{0\} \times \Delta^1}$ and g respectively, it is clear that $(ev_0 \circ p, q)$ detect equivalences.

Then $\mathcal{C} \simeq \text{Fun}(\Delta^1) \times_{\mathcal{C}_0} \mathcal{C}_1$ by [21, Proposition A.8.14]. \square

Corollary 2.13. *With the notation of the previous proposition, for any couple of objects x, y in \mathcal{C} ,*

$$(2.13.1) \quad \text{Map}_{\mathcal{C}}(x, y) \simeq \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C}_0)}(p(x), p(y)) \times_{\text{Map}_{\mathcal{C}_0}(r(x), r(y))} \text{Map}_{\mathcal{C}_1}(q(x), q(y)).$$

Notation 2.14. We will denote by $F_0^{\Delta^1} : \text{Fun}(\Delta^1, \mathcal{C}_0) \rightarrow \text{Fun}(\Delta^1, \mathcal{D}_0)$ (resp. $G_0^{\Delta^1} : \text{Fun}(\Delta^1, \mathcal{D}_0) \rightarrow \text{Fun}(\Delta^1, \mathcal{C}_0)$) the functor corresponding to $(\text{Fun}(\Delta^1, \mathcal{C}_0) \xrightarrow{ev_0} \mathcal{C}_0 \xrightarrow{F_0} \mathcal{D}_0, \text{Fun}(\Delta^1, \mathcal{C}_0) \xrightarrow{ev_1} \mathcal{C}_0 \xrightarrow{F_0} \mathcal{D}_0, F_0 \circ ev_0 \rightarrow F_0 \circ ev_1)$ (resp. $(\text{Fun}(\Delta^1, \mathcal{D}_0) \xrightarrow{ev_0} \mathcal{D}_0 \xrightarrow{G_0} \mathcal{C}_0, \text{Fun}(\Delta^1, \mathcal{D}_0) \xrightarrow{ev_1} \mathcal{D}_0 \xrightarrow{G_0} \mathcal{C}_0, G_0 \circ ev_0 \rightarrow G_0 \circ ev_1)$).

Lemma 2.15. $F_0^{\Delta^1} : \text{Fun}(\Delta^1, \mathcal{C}_0) \rightleftarrows \text{Fun}(\Delta^1, \mathcal{D}_0) : G_0^{\Delta^1}$ are adjoint functors.

Proof. This follows immediately from the definition and from the fact that F_0 is left adjoint to G_0 . \square

Construction 2.16. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be the functor corresponding, by Corollary 2.7, to $(\mathcal{C} \xrightarrow{F_0 \circ L_0} \mathcal{D}_1, \mathcal{C} \xrightarrow{F_1 \circ L_1} \mathcal{D}_1, F_0 \circ L_0 \rightarrow R_0 \circ j_1 \circ F_1 \circ L_1 \simeq F_0 \circ L_0 \circ i_1 \circ L_1)$.

Similarly, let $G : \mathcal{D} \rightarrow \mathcal{C}$ be the functor corresponding, by Corollary 2.7, to $(\mathcal{D} \xrightarrow{G_0 \circ R_0} \mathcal{C}_0, \mathcal{D} \xrightarrow{G_1 \circ R_1} \mathcal{C}_0, G_0 \circ R_0 \rightarrow L_0 \circ i_1 \circ G_1 \circ R_1 \simeq G_0 \circ R_0 \circ j_1 \circ R_1)$.

Let $u : id_{\mathcal{C}} \rightarrow G \circ F$ denote the natural transformation defined as follows. By Proposition 2.6, $\text{Fun}(\mathcal{C}, \mathcal{C})$ is the recollement of $\text{Fun}(\mathcal{C}, \mathcal{C}_0)$ and $\text{Fun}(\mathcal{C}, \mathcal{C}_1)$. By Corollary 2.7, in order to define u we need to exhibit two functors $u_0 : \Delta^1 \rightarrow \text{Fun}(\mathcal{C}, \mathcal{C}_0)$, $u_1 : \Delta^1 \rightarrow \text{Fun}(\mathcal{C}, \mathcal{C}_1)$ and a natural transformation $u_0 \rightarrow L_0 \circ i_1 \circ u_1$. We define

- $u_0 : L_0 \rightarrow L_0 \circ G \circ F \simeq G_0 \circ R_0 \circ F \simeq G_0 \circ F_0 \circ L_0$ induced by $id \rightarrow G_0 \circ F_0$;
- $u_1 : L_1 \rightarrow L_1 \circ G \circ F \simeq G_1 \circ R_1 \circ F \simeq G_1 \circ F_1 \circ L_1$ induced by $id \rightarrow G_1 \circ F_1$;
- $u_0 \rightarrow L_0 \circ i_1 \circ u_1$ as

$$(2.16.1) \quad \begin{array}{ccc} L_0 & \xrightarrow{u_0} & G_0 \circ F_0 \circ L_0 \\ \downarrow & & \downarrow \\ L_0 \circ i_1 \circ L_1 & \xrightarrow{L_0 \circ i_1 \circ u_1} & L_0 \circ i_1 \circ G_1 \circ F_1 \circ L_1 \simeq G_0 \circ R_0 \circ j_1 \circ F_1 \circ L_1, \end{array}$$

where the left vertical arrow is induced by $id \rightarrow i_1 \circ L_1$ and the right vertical arrow by $F_0 \circ L_0 \rightarrow R_0 \circ j_1 \circ F_1 \circ L_1$.

Proposition 2.17. $u : id_{\mathcal{C}} \rightarrow G \circ F$ is a unit transformation for (F, G) (see [20, Definition 5.2.2.7]). In particular, F is a left adjoint to G .

Proof. Let x (resp. y) be an object of \mathcal{C} (resp. \mathcal{D}). We need to show that the morphism

$$(2.17.1) \quad \text{Map}_{\mathcal{D}}(F(x), y) \rightarrow \text{Map}_{\mathcal{C}}(G \circ F(x), G(y)) \xrightarrow{u(x)} \text{Map}_{\mathcal{C}}(x, G(y))$$

is a weak-equivalence.

By Proposition 2.12, we have two pullback squares

$$(2.17.2) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{q_{\mathcal{C}} \sim L_1} & \mathcal{C}_1 \\ \downarrow p_{\mathcal{C}} \sim \overline{sp}_{\mathcal{C}} & \searrow r_{\mathcal{C}} & \downarrow L_0 \circ i_1 \\ \text{Fun}(\Delta^1, \mathcal{C}_0) & \xrightarrow{ev_1} & \mathcal{C}_0 \end{array} \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{q_{\mathcal{D}} \sim R_1} & \mathcal{D}_1 \\ \downarrow p_{\mathcal{D}} \sim \overline{sp}_{\mathcal{D}} & \searrow r_{\mathcal{D}} & \downarrow R_0 \circ j_1 \\ \text{Fun}(\Delta^1, \mathcal{D}_0) & \xrightarrow{ev_1} & \mathcal{D}_0 \end{array}$$

and the map (2.17.1), by Corollary 2.13, corresponds to

$$(2.17.3) \quad \begin{array}{c} \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{D}_0)}(p_{\mathcal{D}} \circ F(x), p_{\mathcal{D}}(y)) \times_{\mathrm{Map}_{\mathcal{D}_0}(r_{\mathcal{D}} \circ F(x), r_{\mathcal{D}}(y))} \mathrm{Map}_{\mathcal{D}_1}(q_{\mathcal{D}} \circ F(x), q_{\mathcal{D}}(y)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{C}_0)}(p_{\mathcal{C}} \circ G \circ F(x), p_{\mathcal{C}} \circ G(y)) \times_{\mathrm{Map}_{\mathcal{C}_0}(r_{\mathcal{C}} \circ G \circ F(x), r_{\mathcal{C}} \circ G(y))} \mathrm{Map}_{\mathcal{C}_1}(q_{\mathcal{C}} \circ G \circ F(x), q_{\mathcal{C}} \circ G(y)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{C}_0)}(p_{\mathcal{C}}(x), p_{\mathcal{C}} \circ G(y)) \times_{\mathrm{Map}_{\mathcal{C}_0}(r_{\mathcal{C}}(x), r_{\mathcal{C}} \circ G(y))} \mathrm{Map}_{\mathcal{C}_1}(q_{\mathcal{C}}(x), q_{\mathcal{C}} \circ G(y)). \end{array}$$

As we have equivalences

$$\begin{array}{ll} \bullet p_{\mathcal{D}} \circ F \simeq F_0^{\Delta^1} \circ p_{\mathcal{C}}, & p_{\mathcal{C}} \circ G \simeq G_0^{\Delta^1} \circ p_{\mathcal{D}}, \\ \bullet q_{\mathcal{D}} \circ F \simeq F_1 \circ L_1, & q_{\mathcal{C}} \circ G \simeq G_1 \circ R_1, \\ \bullet r_{\mathcal{D}} \circ F \simeq R_0 \circ j_1 \circ F_1 \circ L_1, & r_{\mathcal{C}} \circ G \simeq L_0 \circ i_1 \circ G_1 \circ R_1, \end{array}$$

the compositions

$$(2.17.4) \quad \begin{array}{c} \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{D}_0)}(p_{\mathcal{D}} \circ F(x), p_{\mathcal{D}}(y)) \simeq \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{D}_0)}(F_0^{\Delta^1} \circ p_{\mathcal{C}}(x), p_{\mathcal{D}}(y)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{C}_0)}(p_{\mathcal{C}} \circ G \circ F(x), p_{\mathcal{C}} \circ G(y)) \simeq \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{C}_0)}(G_0^{\Delta^1} \circ F_0^{\Delta^1} \circ p_{\mathcal{C}}(x), G_0^{\Delta^1} \circ p_{\mathcal{D}}(y)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{C}_0)}(p_{\mathcal{C}}(x), p_{\mathcal{C}} \circ G(y)) \simeq \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{C}_0)}(p_{\mathcal{C}}(x), G_0^{\Delta^1} \circ p_{\mathcal{D}}(y)) \end{array}$$

and

$$(2.17.5) \quad \begin{array}{c} \mathrm{Map}_{\mathcal{D}_1}(q_{\mathcal{D}} \circ F(x), q_{\mathcal{D}}(y)) \simeq \mathrm{Map}_{\mathcal{D}_1}(F_1 \circ L_1(x), q_{\mathcal{D}}(y)) = \mathrm{Map}_{\mathcal{D}_1}(F_1 \circ L_1(x), R_1(y)) \\ \downarrow \\ \mathrm{Map}_{\mathcal{C}_1}(q_{\mathcal{C}} \circ G \circ F(x), q_{\mathcal{C}} \circ G(y)) \simeq \mathrm{Map}_{\mathcal{C}_1}(G_1 \circ F_1 \circ L_1(x), G_1 \circ R_1(y)) \\ \downarrow \\ \mathrm{Map}_{\mathcal{C}_1}(q_{\mathcal{C}}(x), q_{\mathcal{C}} \circ G(y)) \simeq \mathrm{Map}_{\mathcal{C}_1}(L_1(x), G_1 \circ R_1(y)) \end{array}$$

are weak-equivalences, as $(F_0^{\Delta^1}, G_0^{\Delta^1})$ and (F_1, G_1) are adjoint functors. Similarly, the composition

$$(2.17.6) \quad \begin{array}{c} \mathrm{Map}_{\mathcal{D}_0}(r_{\mathcal{D}} \circ F(x), r_{\mathcal{D}}(y)) \simeq \mathrm{Map}_{\mathcal{D}_0}(F_0 \circ L_0 \circ i_1 \circ L_1(x), R_0 \circ j_1 \circ R_1(y)) \\ \downarrow \\ \mathrm{Map}_{\mathcal{C}_0}(r_{\mathcal{C}} \circ G \circ F(x), r_{\mathcal{C}} \circ G(y)) \simeq \mathrm{Map}_{\mathcal{D}_0}(G_0 \circ F_0 \circ L_0 \circ i_1 \circ L_1(x), G_0 \circ R_0 \circ j_1 \circ R_1(y)) \\ \downarrow \\ \mathrm{Map}_{\mathcal{C}_0}(r_{\mathcal{C}}(x), r_{\mathcal{C}} \circ G(y)) \simeq \mathrm{Map}_{\mathcal{D}_0}(L_0 \circ i_1 \circ L_1(x), G_0 \circ R_0 \circ j_1 \circ R_1(y)) \end{array}$$

is a weak-equivalence as (F_0, G_0) is an adjunction.

Then (2.17.1) is a weak-equivalence and the lemma follows from [20, Proposition 5.2.2.8]. \square

3. REVIEW OF THE THEORY OF TAME VANISHING CYCLES

In this section we will review the theory of (tame) vanishing cycles as developed in [10, Exposé II], [11, Exposé XIII].

The classical definition of tame vanishing cycles. Let A be an henselian discrete valuation ring. Fix an uniformiser π . Then the residue field of A is $k := A/(\pi)$, while the fraction field of A is $K := A[\pi^{-1}]$. Let p denote the characteristic of k . Fix a separable closure k^s of k and a separable closure K^s of K . Let K^u and K^t denote the maximal unramified and maximal tamely ramified extensions of K inside K^s . We have a short exact sequence of profinite groups (see [26])

$$(3.0.1) \quad 1 \rightarrow I^t := \text{Gal}(K^t/K^u) \rightarrow \text{Gal}(K^t/K) \rightarrow \text{Gal}(K^u/K) \simeq \text{Gal}(k^s/k) \rightarrow 1.$$

I^t is called the *tame inertia group*. There is an isomorphism

$$(3.0.2) \quad I^t \simeq \varprojlim_{(n,p)=1} \mu_n(K^s),$$

where $\mu_n(K^s)$ denotes the group of n^{th} roots of the unity in K^s .

We will use the following notation:

- $S := \text{Spec}(A)$,
- $\sigma := \text{Spec}(k)$,
- $\sigma^s := \text{Spec}(k^s)$,
- $\eta := \text{Spec}(K)$,
- $\eta^u := \text{Spec}(K^u)$,
- $\eta^t := \text{Spec}(K^t)$.

For an S -scheme $p : X \rightarrow S$, we will consider the following diagram in which all squares are Cartesian:

$$(3.0.3) \quad \begin{array}{ccccccc} & & & & & & j^t \\ & & & & & & \downarrow \\ & & & & & & \downarrow \\ & & & & & & \downarrow \\ X_{\sigma^s} & \xrightarrow{i^s} & X_{\sigma} & \xrightarrow{i} & X & \xleftarrow{j^u} & X_{\eta^u} & \xleftarrow{j} & X_{\eta} & \xleftarrow{j^t} & X_{\eta^t} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \sigma^s & \xrightarrow{i_0^s} & \sigma & \xrightarrow{i_0} & S & \xleftarrow{j_0^u} & \eta^u & \xleftarrow{j_0} & \eta & \xleftarrow{j_0^t} & \eta^t \\ & & & & & & & & & & \downarrow \\ & & & & & & & & & & j_0^t \end{array}$$

Let $\mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)}$ denote the ∞ -category of étale sheaves of Λ -modules on X_{σ^s} endowed with a continuous action of $\text{Gal}(k^s/k)$ compatible with the action on X_{σ^s} . Similarly, let $\mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(K^t/K)}$ denote the ∞ -category of sheaves on X_{σ^s} endowed with a continuous action of $\text{Gal}(K^t/K)$ compatible with the action on X_{σ^s} induced by the morphism $\text{Gal}(K^t/K) \rightarrow \text{Gal}(k^s/k)$. These two categories are related by an adjunction:

$$(3.0.4) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)} \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(K^t/K)} : (-)^{I^t}.$$

Roughly, for $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)}$, $\text{triv}(\mathcal{F})$ is the sheaf on X_{σ^s} with the continuous $\text{Gal}(K^t/K)$ -action induced by $\text{Gal}(K^t/K) \rightarrow \text{Gal}(k^s/k)$, while for $\mathcal{G} \in \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(K^t/K)}$, \mathcal{G}^{I^t} is the sheaf of I^t -homotopy fixed points of \mathcal{G} , with the induced $\text{Gal}(k^s/k)$ -action.

Notation 3.1. With a little abuse of notation, for $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)}$, we will write \mathcal{F} instead of $\text{triv}(\mathcal{F})$. We hope that it will be clear from the context whether we are considering one object or the other.

For $\mathcal{G} \in \mathcal{D}_{\text{ét}}(X_\eta; \Lambda)$, the sheaf of *tame nearby cycles with coefficients in \mathcal{G}* is

$$(3.1.1) \quad \Psi_{\eta,p}^{cl,t}(\mathcal{G}) := (i^s)^* \circ (j^t)_*(\mathcal{G}|_{X_{\eta^t}}) \in \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}.$$

The $Gal(K^t/K)$ -action on $\Psi_{\eta,p}^{cl,t}(\mathcal{G})$ is the one induced by that on $\mathcal{G}|_{X_{\eta^t}} \in \mathcal{D}_{\text{ét}}(X_{\eta^t}; \Lambda)^{Gal(K^t/K)}$.

For $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X; \Lambda)$, the unit of the adjunction $((j^t)^*, (j^t)_*)$ induces a morphism

$$(3.1.2) \quad sp : (i^s)^*(\mathcal{F}) \rightarrow \Psi_{\eta,p}^{cl,t}(j^*\mathcal{F})$$

in the ∞ -category $\mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}$. This morphism is called the *specialization morphism* and its cone, denoted by

$$(3.1.3) \quad \Phi_p^{cl,t}(\mathcal{F}) \in \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)},$$

is called the sheaf of *tame vanishing cycles with coefficients in \mathcal{F}* .

Tame vanishing cycles via Deligne-Mumford stacks. Our aim in this subsection will be to reinterpret the definition of nearby and vanishing cycles using the language of pro Deligne-Mumford stacks. Indeed, the ∞ -categories $\mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(k^s/k)}$ and $\mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}$ can (and will) be interpreted as the ∞ -categories of étale sheaves of Λ -modules on some pro Deligne-Mumford stacks $[X_{\sigma^s}/Gal(k^s/k)]$ and $[X_{\sigma^s}/Gal(K^t/K)]$. The reason why we need to consider pro Deligne-Mumford stacks is that the groups $Gal(k^s/k)$ and $Gal(K^t/K)$ are profinite.

The pro DM stack $[X_{\sigma^s}/Gal(k^s/k)]$. Write k^s as the colimit $\varinjlim_i k_i$, where $k \subseteq k_i$ runs in the filtered set \mathcal{I} of finite Galois extensions of k inside k^s .

For every finite Galois extension $k \subseteq k_i \subseteq k^s$, let $\sigma_i := Spec(k_i)$ and let $S_i \rightarrow S$ be the corresponding finite étale covering of S . Put $S^s = \varprojlim_i S_i$ and set $X_{S_i} = X \times_S S_i$, $X_{S^s} = \varprojlim_i X_{S_i}$. Moreover, put $X_{\sigma_i} := X_\sigma \times_X X_{S_i}$. The group $Aut_S(S_i) \simeq Gal(k_i/k)$ acts naturally on X_{S_i} and on X_{σ_i} .

Remark 3.2. Notice that for each $i \in \mathcal{I}$, X_{σ_i} is a nilthickening of $X_\sigma \times_{Spec(\sigma)} Spec(\sigma_i)$.

For every tower of finite Galois extensions $k \subseteq k_i \subseteq k_j$ inside k^s , $Gal(k_j/k)$ acts on X_{σ_i} by means of the canonical quotient morphism

$$(3.2.1) \quad Gal(k_j/k) \rightarrow Gal(k_j/k)/Gal(k_j/k_i) \simeq Gal(k_i/k).$$

In particular, we have a diagram of Deligne-Mumford stacks

$$(3.2.2) \quad \begin{array}{c} \mathcal{I}^{op} \rightarrow \mathbf{DM}_S \\ i \leq j \mapsto [X_{\sigma_j}/Gal(k_j/k)] \xrightarrow{\alpha_{ij}} [X_{\sigma_i}/Gal(k_i/k)]. \end{array}$$

Notice that this is the diagram obtained from the diagram

$$(3.2.3) \quad \begin{array}{c} \mathcal{I}^{op} \rightarrow \mathbf{DM}_S \\ i \leq j \mapsto [X_{S_j}/Gal(k_j/k)] \rightarrow [X_{S_i}/Gal(k_i/k)] \end{array}$$

by pullbacks along $X_\sigma \rightarrow X$.

Notation 3.3. For every finite Galois extension $k \subseteq k_i$, we will use the following notation:

$$(3.3.1) \quad \mathcal{D}_{\text{ét}}(X_{\sigma_i}; \Lambda)^{Gal(k_i/k)} := \mathcal{D}_{\text{ét}}([X_{\sigma_i}/Gal(k_i/k)]; \Lambda).$$

For every tower of finite Galois extensions $k \subseteq k_i \subseteq k_j$, the adjunction

$$(3.3.2) \quad \alpha_{ij}^* : \mathcal{D}_{\text{ét}}([X_{\sigma_i}/\text{Gal}(k_i/k)]; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}([X_{\sigma_j}/\text{Gal}(k_j/k)]; \Lambda) : (\alpha_{ij})_*$$

identifies with

$$v_{ij}^* : \mathcal{D}_{\text{ét}}(X_{\sigma_i}; \Lambda)^{\text{Gal}(k_i/k)} \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma_j}; \Lambda)^{\text{Gal}(k_j/k)} : (v_{ij})_* \circ (-)^{\text{Gal}(k_j/k_i)},$$

where $v_{ij} : X_{\sigma_j} \rightarrow X_{\sigma_i}$ is the morphism induced by $S_j \rightarrow S_i$. We define

$$(3.3.3) \quad [X_{\sigma^s}/\text{Gal}(k^s/k)] := \varprojlim_i [X_{\sigma_i}/\text{Gal}(k_i/k)],$$

the pro-object in the 2-category of DM stacks corresponding to the diagram (3.2.2).

Definition 3.4. Consider the ∞ -functor

$$(3.4.1) \quad \mathbf{DM}_S \rightarrow \mathbf{Pr}_{\text{stb}}^{\mathbf{R}}$$

which expresses the functoriality of the assignments

$$f : \mathfrak{Y} \rightarrow \mathfrak{X} \mapsto f_* : \mathcal{D}_{\text{ét}}(\mathfrak{Y}; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathfrak{X}; \Lambda).$$

We define the ∞ -category of $\text{Gal}(k^s/k)$ -equivariant sheaves on X_{σ^s} as the limit of the diagram

$$(3.4.1) \circ (3.2.2) : \mathcal{S}^{op} \rightarrow \mathbf{DM}_S \rightarrow \mathbf{Pr}_{\text{stb}}^{\mathbf{R}},$$

i.e. as

$$(3.4.2) \quad \begin{aligned} \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)} &= \mathcal{D}_{\text{ét}}([X_{\sigma^s}/\text{Gal}(k^s/k)]; \Lambda) := \\ &= \varprojlim_i \mathcal{D}_{\text{ét}}([X_{\sigma_i}/\text{Gal}(k_i/k)]; \Lambda) = \varprojlim_i \mathcal{D}_{\text{ét}}(X_{\sigma_i}; \Lambda)^{\text{Gal}(k_i/k)} \in \mathbf{Pr}_{\text{stb}}^{\mathbf{R}}, \end{aligned}$$

Remark 3.5. For every finite Galois extension $k \subseteq k_i \subseteq k^s$, let $\alpha_i : [X_{\sigma_i}/\text{Gal}(k_i/k)] \rightarrow X_{\sigma}$ denote the canonical morphism. It is a homeomorphism. The adjunction

$$(3.5.1) \quad \alpha_i^* : \mathcal{D}_{\text{ét}}(X_{\sigma}; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}([X_{\sigma_i}/\text{Gal}(k_i/k)]; \Lambda) : (\alpha_i)_*$$

identifies with the adjunction

$$(3.5.2) \quad v_i^* : \mathcal{D}_{\text{ét}}(X_{\sigma}; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma_i}; \Lambda)^{\text{Gal}(k_i/k)} : (v_i)_* \circ (-)^{\text{Gal}(k_i/k)},$$

where $v_i : X_{\sigma_i} \rightarrow X_{\sigma}$. These functors are quasi-inverses.

Taking the limit over all i 's¹, we get the well known equivalence [11, Exposé XIII]

$$(3.5.3) \quad v^* : \mathcal{D}_{\text{ét}}(X_{\sigma}; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)} : v_* \circ (-)^{\text{Gal}(k^s/k)},$$

where $v : X_{\sigma^s} \rightarrow X_{\sigma}$.

One can think of the homomorphism $[X_{\sigma^s}/\text{Gal}(k^s/k)] \simeq X_{\sigma}$ as the reason why there is an equivalence

$$(3.5.4) \quad \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(k^s/k)} \simeq \mathcal{D}_{\text{ét}}(X_{\sigma}; \Lambda).$$

¹Technically, to take this limit we have to produce a diagram $\mathcal{S}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\mathbf{R}})$, $i \mapsto (\phi_i)_*$, but we won't give the (easy) details of this construction here.

The pro DM stack $[X_{\sigma^s}/\text{Gal}(K^t/K)]$. Write K^t as the colimit $\varinjlim_a K_a$, where a runs in the filtered set \mathcal{A} of finite Galois extensions $K \subseteq K_a \subseteq K^t$. For such an extension, denote by $K \subseteq K_a^u \subseteq K_a$ the maximal unramified extension of K contained in K_a . We will denote by $k \subseteq k_a$ (resp. $S_a \rightarrow S$) the finite Galois extension (resp. finite étale covering) of k (resp. S) corresponding to $K \subseteq K_a^u$ (see e.g. [26]). This provides a morphism of filtered sets

$$(3.5.5) \quad \mathcal{A} \rightarrow \mathcal{I}.$$

The canonical morphism

$$(3.5.6) \quad \text{Gal}(K_a/K) \rightarrow \text{Gal}(K_a/K)/\text{Gal}(K_a^u/K) \simeq \text{Gal}(K_a^u/K) \simeq \text{Gal}(k_a/k)$$

endows X_{σ_a} with an action of $\text{Gal}(K_a/K)$.

Every tower of finite Galois extensions $K \subseteq K_a \subseteq K_b$ inside K^t induces a tower of finite Galois extensions $K \subseteq K_a^u \subseteq K_b^u$ inside K^u and therefore a commutative square

$$(3.5.7) \quad \begin{array}{ccc} \text{Gal}(K_b/K) & \longrightarrow & \text{Gal}(k_b/k) \\ \downarrow & & \downarrow \\ \text{Gal}(K_a/K) & \longrightarrow & \text{Gal}(k_a/k) \end{array}$$

which endows X_{σ_a} with an action of $\text{Gal}(K_b/K)$.

In particular, we have a diagram of Deligne-Mumford stacks

$$(3.5.8) \quad \begin{array}{c} \mathcal{A}^{op} \rightarrow \mathbf{DM}_S \\ a \leq b \mapsto [X_{\sigma_b}/\text{Gal}(K_b/K)] \xrightarrow{\beta_{ab}} [X_{\sigma_a}/\text{Gal}(K_a/K)]. \end{array}$$

Notation 3.6. For every finite Galois extension $K \subseteq K_a \subseteq K^t$, we will use the following notation:

$$(3.6.1) \quad \mathcal{D}_{\text{ét}}(X_{\sigma_a}; \Lambda)^{\text{Gal}(K_a/K)} := \mathcal{D}_{\text{ét}}([X_{\sigma_a}/\text{Gal}(K_a/K)]; \Lambda).$$

For every tower of finite Galois extensions $K \subseteq K_a \subseteq K_b$, the adjunction

$$(3.6.2) \quad \beta_{ab}^* : \mathcal{D}_{\text{ét}}([X_{\sigma_a}/\text{Gal}(K_a/K)]; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}([X_{\sigma_b}/\text{Gal}(K_b/K)]; \Lambda) : (\beta_{ab})_*$$

identifies with

$$u_{ab}^* : \mathcal{D}_{\text{ét}}(X_{\sigma_a}; \Lambda)^{\text{Gal}(K_a/K)} \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma_b}; \Lambda)^{\text{Gal}(K_b/K)} : (u_{ab})_* \circ (-)^{\text{Gal}(K_b/K_a)},$$

where $u_{ab} : X_{\sigma_b} \rightarrow X_{\sigma_a}$ is the morphism induced by $S_b \rightarrow S_a$.

We define the pro Deligne-Mumford stack

$$(3.6.3) \quad [X_{\sigma^s}/\text{Gal}(K^t/K)] := \varprojlim_a [X_{\sigma_a}/\text{Gal}(K_a/K)].$$

Definition 3.7. We define the ∞ -category of $\text{Gal}(K^t/K)$ -equivariant sheaves on X_{σ^s} as the limit of the diagram

$$(3.7.1) \quad (3.4.1) \circ (3.5.8) : \mathcal{A}^{op} \rightarrow \mathbf{Pr}_{\text{stb}}^{\mathbf{R}},$$

i.e. as

$$(3.7.2) \quad \begin{aligned} \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{\text{Gal}(K^t/K)} &= \mathcal{D}_{\text{ét}}([X_{\sigma^s}/\text{Gal}(K^t/K)]; \Lambda) := \\ \varprojlim_a \mathcal{D}_{\text{ét}}([X_{\sigma_a}/\text{Gal}(K_a/K)]; \Lambda) &= \varprojlim_a \mathcal{D}_{\text{ét}}(X_{\sigma_a}; \Lambda)^{\text{Gal}(K_a/K)} \in \mathbf{Pr}_{\text{stb}}^{\mathbf{R}}. \end{aligned}$$

Tame vanishing cycles (reprise). For every finite Galois extension $K \subseteq K_a \subseteq K^t$, the morphism of Deligne-Mumford stacks

$$(3.7.3) \quad \chi_a : [X_{\sigma_a}/Gal(K_a/K)] \rightarrow [X_{\sigma_a}/Gal(k_a/k)]$$

induces an adjunction

$$(3.7.4) \quad \chi_a^* : \mathcal{D}_{\text{ét}}([X_{\sigma_a}/Gal(k_a/k)]; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}([X_{\sigma_a}/Gal(K_a/K)]; \Lambda) : (\chi_a)_*$$

which identifies with

$$(3.7.5) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X_{\sigma_a}; \Lambda)^{Gal(k_a/k)} \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma_a}; \Lambda)^{Gal(K_a/K)} : (-)^{Gal(K_a/K_a^u)}.$$

Notice that, by the commutativity of diagram 3.5.7, the morphisms χ_a assemble in a morphism of filtered diagrams of DM stacks

$$(3.7.6) \quad \chi : \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{DM}_S).$$

If we take the limit of the diagram

$$(3.7.7) \quad (3.4.1) \circ (3.7.6) : \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R),$$

we get the adjunction

$$(3.7.8) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(k^s/k)} \rightleftarrows \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)} : (-)^{I^t}.$$

Definition 3.8. With the same notation as above, we define $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ to be the recollement determined by the adjunction 3.7.8.

We will denote by

$$(3.8.1) \quad (i^s)^* : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(k^s/k)}$$

the ∞ -functor

$$(3.8.2) \quad \mathcal{D}_{\text{ét}}(X; \Lambda) \xrightarrow{i^*} \mathcal{D}_{\text{ét}}(X_{\sigma}; \Lambda) \xrightarrow{v^* \simeq} \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(k^s/k)}.$$

Next we redefine the ∞ -functor of tame nearby cycles.

Construction 3.9. As a first step, we define an ∞ -functor

$$(3.9.1) \quad \bar{j}_* : \mathcal{D}_{\text{ét}}(X_{\eta^t}; \Lambda)^{Gal(K^t/K)} \rightarrow \mathcal{D}_{\text{ét}}(X_{S^s}, \Lambda)^{Gal(K^t/K)}.$$

Let $\eta_a := \text{Spec}(K_a)$ and let $X_{\eta_a} = X_{\eta} \times_{\eta} \eta_a$. Then $Gal(K_a/K)$ acts on X_{η_a} .

We have that

$$(3.9.2) \quad \text{Aut}_S(S_a) \simeq Gal(k_a/k).$$

In particular, $Gal(K_a/K)$ acts on $X_{S_a} = X \times_S S_a$ by means of the morphism $Gal(K_a/K) \rightarrow \text{Aut}_S(S_a) \simeq Gal(k_a/k)$. This gives us a $Gal(K_a/K)$ -equivariant map

$$(3.9.3) \quad X_{\eta_a} \xrightarrow{j_a} X_{S_a}$$

which defines a morphism of DM stacks

$$(3.9.4) \quad [X_{\eta_a}/Gal(K_a/K)] \xrightarrow{\bar{j}_a} [X_{S_a}/Gal(K_a/K)].$$

These morphisms assemble in a diagram

$$(3.9.5) \quad \bar{j} : \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{DM}_S).$$

The ∞ -functor 3.4.1 induces a diagram

$$(3.9.6) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}})$$

and we define 3.9.1 as its limit.

Remark 3.10. Notice that, as $[X_{\eta_a}/\text{Gal}(K_a/K)] \simeq X_{\eta}$, the source of this ∞ -functor is equivalent to $\mathcal{D}_{\text{ét}}(X_{\eta}; \Lambda)$. Also notice that if a is associated to a finite, unramified Galois extension of K , then we also have that $[X_{S_a}/\text{Gal}(K_a/K)] \simeq X$. In particular, if we restrict the above diagram to the category \mathcal{A}^u corresponding to unramified extensions, we get an ∞ -functor

$$(3.10.1) \quad j_*^u : \mathcal{D}_{\text{ét}}(X_{\eta^u}; \Lambda)^{\text{Gal}(K^u/K)} \rightarrow \mathcal{D}_{\text{ét}}(X_{S^s}; \Lambda)^{\text{Gal}(k^s/k)}$$

that is canonically identified with

$$(3.10.2) \quad j_* : \mathcal{D}_{\text{ét}}(X_{\eta}; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X; \Lambda).$$

The next step will be to define the pullback

$$(3.10.3) \quad \bar{i}^* : \mathcal{D}_{\text{ét}}(X_{S^s}, \Lambda)^{\text{Gal}(K^t/K)} \rightarrow \mathcal{D}_{\text{ét}}(X_{\sigma^s}, \Lambda)^{\text{Gal}(K^t/K)}.$$

Here $X_{S^s} = \varprojlim_{a \in \mathcal{A}} X_{S_a}$. Notice that, for every $a \in \mathcal{A}$, there is a $\text{Gal}(K_a/K)$ -equivariant map

$$(3.10.4) \quad X_{\sigma_a} \xrightarrow{i_a} X_{S_a}$$

which induces a diagram

$$(3.10.5) \quad \bar{i} : \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{DM}_S).$$

Remark 3.11. At this point, one could consider the limit $\mathcal{D}_{\text{ét}}(X_{\sigma^s}, \Lambda)^{\text{Gal}(K^t/K)} \rightarrow \mathcal{D}_{\text{ét}}(X_{S^s}, \Lambda)^{\text{Gal}(K^t/K)}$ of the diagram

$$(3.4.1) \circ (3.10.5) : \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}})$$

and define 3.10.3 as its left adjoint. However, with this definition it would not be clear that we have identities

$$(-)^{\text{Gal}(K^t/K_a)} \circ \bar{i}^* \simeq i_a^* \circ (-)^{\text{Gal}(K^t/K_a)}.$$

Therefore, we will give another (equivalent) definition, for which the above equivalences will be tautological.

Construction 3.12. Consider the diagram $\bar{X} : \mathcal{A}^{op} \rightarrow \mathbf{DM}_S$ given by $a \mapsto [X_{S_a}/\text{Gal}(K_a/K)]$. Also consider the closed embedding $X_{\sigma} \hookrightarrow X$. Then we can use the construction of Appendix A and get a diagram

$$(3.12.1) \quad N(\mathcal{A}^{op}) \rightarrow \text{Fun}(\Delta^1, \text{Corr}(N(\mathbf{DM}_S))_{F, \text{all}}).$$

Indeed, since we put $X_{\sigma_i} = X_{S_i} \times_X X_{\sigma}$, we have that for each $a \leq b$

$$(3.12.2) \quad [X_{\sigma_i}/\text{Gal}(K_b/K)] \simeq [X_{\sigma_a}/\text{Gal}(K_a/K)] \times_{[X_{S_a}/\text{Gal}(K_a/K)]} [X_{S_b}/\text{Gal}(K_b/K)].$$

We can compose it with the ∞ -functor 1.0.27 and get a diagram

$$(3.12.3) \quad N(\mathcal{A}^{op}) \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{L}}).$$

Finally, using the equivalences $\mathbf{Pr}_{\text{stb}}^{\text{L}} \simeq (\mathbf{Pr}_{\text{stb}}^{\text{R}})^{op}$ and $\Delta^1 \simeq (\Delta^1)^{op}$ we get another diagram

$$(3.12.4) \quad N(\mathcal{A}^{op}) \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}})$$

and define 3.10.3 as its limit.

Remark 3.13. Notice that it now follows from our definition that we have (natural) equivalences

$$(-)^{Gal(K^t/K_a)} \circ \bar{i}^* \simeq i_a^* \circ (-)^{Gal(K^t/K_a)},$$

as $(-)^{Gal(K^t/K_a)} : \mathcal{D}_{\acute{e}t}(X_{S^s}; \Lambda)^{Gal(K^t/K)} = \varprojlim \mathcal{D}_{\acute{e}t}(X_{S_b})^{Gal(K_b/K)} \rightarrow \mathcal{D}_{\acute{e}t}(X_{S_a}; \Lambda)^{Gal(K_a/K)}$ (resp. $(-)^{Gal(K^t/K_a)} : \mathcal{D}_{\acute{e}t}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)} = \varprojlim \mathcal{D}_{\acute{e}t}(X_{\sigma_b})^{Gal(K_b/K)} \rightarrow \mathcal{D}_{\acute{e}t}(X_{\sigma_a}; \Lambda)^{Gal(K_a/K)}$) is the projection onto the a^{th} factor.

Remark 3.14. The construction can be performed in the language of [12] as well. Consider the ∞ -functor 3.4.1. We will now use some notation and terminology borrowed from [12]. As every morphism of DM stacks is representable by DM stacks, it follows from the results recalled in Section 1 that for every proper morphism $f : \mathfrak{Y} \rightarrow \mathfrak{X}$, the ∞ -functor $f_* : \mathcal{D}_{\acute{e}t}(\mathfrak{Y}; \Lambda) \rightarrow \mathcal{D}_{\acute{e}t}(\mathfrak{X}; \Lambda)$ admits a left adjoint. Moreover, (3.4.1) satisfies the left Beck-Chevalley condition with respect to *prop*, the class of proper morphisms. By [12, Theorem 3.2.2, Chapter 7], this implies that we have a functor

$$(3.14.1) \quad Corr(\mathbf{DM}_S)_{all,prop}^{prop} \rightarrow (\infty, 2)\mathbf{Pr}_{stb}^R.$$

We shall restrict it to

$$(3.14.2) \quad Corr(\mathbf{DM}_S)_{all,prop}^{pceq} \rightarrow \mathbf{Pr}_{stb}^R,$$

where $pceq \subseteq prop$ denotes the class of proper morphism of DM stacks such that the induced pushforward is an equivalence. At the level of objects, this ∞ -functor sends a DM stack \mathfrak{X} to the ∞ -category $\mathcal{D}_{\acute{e}t}(\mathfrak{X}; \Lambda)$. At the level of 1-morphisms, it sends a correspondence

$$\begin{array}{c} \mathfrak{X} \xrightarrow{f} \mathfrak{Y} \\ \downarrow g \\ \mathfrak{Z} \end{array}$$

to the ∞ -functor $g_* \circ f^! : \mathcal{D}_{\acute{e}t}(\mathfrak{Y}; \Lambda) \rightarrow \mathcal{D}_{\acute{e}t}(\mathfrak{Z}; \Lambda)$.

We will now need to define a diagram

$$(3.14.3) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, Corr(\mathbf{DM}_S)_{all,prop}^{pceq}).$$

In order to do so, compose the ∞ -functors $(\mathbf{DM}_S)_{prop} \rightarrow (\mathbf{DM}_S)_{all} \rightarrow Corr(\mathbf{DM}_S)_{all,prop}^{prop}$ with the diagram

$$(3.14.4) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{DM}_S)$$

induced by the morphisms $X_\sigma \times_\sigma \sigma_a \rightarrow X_{S_a}$. This way, we get a diagram

$$(3.14.5) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, Corr(\mathbf{DM}_S)_{all,prop}^{prop}).$$

For $a \leq b$ in \mathcal{A} , the associated morphism of correspondences of DM stacks is

$$(3.14.6) \quad \begin{array}{ccc} \bar{X}_{\sigma_b} & \xrightarrow{\beta_{ab}} & \bar{X}_{\sigma_a} \\ \searrow & \downarrow & \downarrow i_a \\ \bar{X}_{\sigma_a} \times_{\bar{X}_{S_a}} \bar{X}_{S_b} & \xrightarrow{\beta_{ab}} & \bar{X}_{\sigma_a} \\ \downarrow i_b & & \downarrow \\ \bar{X}_{S_b} & \xrightarrow{\beta_{ab}} & \bar{X}_{S_a} \end{array}$$

where $\bar{X}_{S_a} = [X_{S_a}/Gal(K_a/K)]$ (resp. ...). In particular, as the morphisms $\bar{X}_{\sigma_b} \rightarrow \bar{X}_{\sigma_a} \times_{\bar{X}_{S_a}} \bar{X}_{S_b}$ are nilthickenings, we have an induced diagram

$$(3.14.7) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \text{Corr}(\mathbf{DM}_S)^{pceq}_{all,prop}).$$

By composing it with 3.14.2, we get a diagram

$$(3.14.8) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R).$$

Then diagrams 3.14.6 take care of the functoriality (in a, b) of the natural equivalences

$$(i_b)_* \circ (\beta_{ab})^! \simeq (\beta_{ab})^! \circ (i_a)_*$$

Finally, using the equivalences $\mathbf{Pr}_{\text{stb}}^R \simeq (\mathbf{Pr}_{\text{stb}}^L)^{op}$ and $\Delta^1 \simeq (\Delta^1)^{op}$, we get a diagram

$$(3.14.9) \quad \mathcal{A}^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^L).$$

This expresses the functoriality (in a, b) of the equivalences

$$(-)^{Gal(K_b/K_a)} \circ i_a^* \simeq i_b^* \circ (-)^{Gal(K_b/K_a)}.$$

We can define 3.10.3 as the limit of this diagram.

Notice that, as both the inclusions $\mathbf{Pr}_{\text{stb}}^L \subseteq \widehat{Cat}_\infty$ and $\mathbf{Pr}_{\text{stb}}^R \subseteq \widehat{Cat}_\infty$ preserve (small) limits, the source (resp. target) of this ∞ -functor is indeed $\mathcal{D}_{\text{ét}}(X_{S^s}; \Lambda)^{Gal(K^t/K)}$ (resp. $\mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}$).

We are finally able to define the *tame nearby cycles functor* $\Psi_{p,\eta}^t : \mathcal{D}_{\text{ét}}(X_\eta; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}$ as the composition

$$(3.14.10) \quad \mathcal{D}_{\text{ét}}(X_\eta; \Lambda) \simeq \mathcal{D}_{\text{ét}}(X_{\eta^t}; \Lambda)^{Gal(K^t/K)} \xrightarrow{\bar{j}_*} \mathcal{D}_{\text{ét}}(X_{S^s}; \Lambda)^{Gal(K^t/K)} \xrightarrow{\bar{i}^*} \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}.$$

It follows immediately from the construction of \bar{j}_* and \bar{i}^* that

$$(3.14.11) \quad (-)^{I^t} \circ \Psi_{p,\eta}^t \simeq (i^s)^* \circ (j^u)_* \circ (j^u)^*.$$

In particular, there is a natural transformation

$$(3.14.12) \quad (i^s)^* \rightarrow (-)^{I^t} \circ \Psi_{p,\eta}$$

induced by the unit of the adjunction $((j^u)^*, (j^u)_*)$. By Corollary 2.7, these data determine an ∞ -functor

$$(3.14.13) \quad \Psi_p^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda),$$

the *total tame vanishing cycles functor*. We recover the *tame vanishing cycles functor*

$$(3.14.14) \quad \Phi_p^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^s/K)}$$

by applying Construction 2.8, i.e. as the composition

$$(3.14.15) \quad \mathcal{D}_{\text{ét}}(X; \Lambda) \xrightarrow{\Psi_p^t} \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \xrightarrow{sp} \text{Fun}(\Delta^1, \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}) \xrightarrow{\text{cofiber}} \mathcal{D}_{\text{ét}}(X_{\sigma^s}; \Lambda)^{Gal(K^t/K)}.$$

We conclude by verifying that the construction we described above agrees with the usual one.

Proposition 3.15. *The ∞ -functors $\Psi_{p,\eta}^{cl,t}$ (resp. $\Phi_p^{cl,t}$) and $\Psi_{p,\eta}$ (resp. Φ_p^t) are equivalent.*

Proof. We obtain a canonical natural transformations $\Psi_{p,\eta}^{cl,t} \rightarrow \Psi_{p,\eta}^t$ (resp. $\Phi_p^{cl,t} \rightarrow \Phi_p^t$) by the canonical equivalences

$$(3.15.1) \quad \mathcal{D}_{\acute{e}t}(X_{\sigma_a}; \Lambda)^{Gal(K_a/K)} \simeq \mathcal{D}_{\acute{e}t}([X_{\sigma_a}/Gal(K_a/K)]; \Lambda) \quad (\text{resp. } \dots)$$

Let $\mathcal{F} \in \mathcal{D}_{\acute{e}t}(X; \Lambda)$. Write $K^t = \varinjlim_a K_a$, where a runs in the filtered set \mathcal{A} of finite Galois extensions $K \subseteq K_a \subseteq K^t$. Consider the diagrams

$$(3.15.2) \quad X_{\sigma_a} \xrightarrow{i_a} X_{S_a} \xleftarrow{j_a} X_{\eta_a}.$$

Then $\Psi_{p,\eta}^{cl,t}(\mathcal{F}) \simeq \varinjlim_a i_a^* \circ (j_a)_*(\mathcal{F}_a)$, where $\mathcal{F}_a = (X_{S_a} \rightarrow X)^*\mathcal{F}$. This is an expression of the fact that $(\Psi_{p,\eta}^{cl,t}(\mathcal{F}))^{Gal(K^t/K_a)} \simeq i_a^* \circ (j_a)_*(\mathcal{F}_a)$ and that the action of $Gal(K^t/K)$ is continuous. Since $i_a^* \circ (j_a)_*$ identifies with $\bar{i}_a^* \circ (\bar{j}_a)_*$, to prove the first claim we are left to show that $\Psi_{p,\eta}^t(\mathcal{F}) \simeq \varinjlim_a \bar{i}_a^* \circ (\bar{j}_a)_*(\mathcal{F}_a)$. By Lemma 9.3, we only need to show that

$$(3.15.3) \quad (\Psi_{p,\eta}^t(\mathcal{G}))^{Gal(K^t/K_a)} \simeq \bar{i}_a^* \circ (\bar{j}_a)_*(\mathcal{F}_a),$$

which follows immediately from the definitions of \bar{i}_a^* and \bar{j}_a^* . The second claim follows from

$$(3.15.4) \quad \Phi_p^{cl,t}(\mathcal{F}) \simeq \varinjlim_a \left(\text{cofib}(i_a^*(\mathcal{F}_a) \rightarrow i_a^* \circ (j_a)_* \circ j_a^*(\mathcal{F}_a)) \right)$$

$$(3.15.5) \quad \simeq \varinjlim_a \left(\text{cofib}(\bar{i}_a^*(\mathcal{F}_a) \rightarrow \bar{i}_a^* \circ (\bar{j}_a)_* \circ \bar{j}_a^*(\mathcal{F}_a)) \right) \simeq \Phi_p^t(\mathcal{F}).$$

□

4. TAME NEARBY AND VANISHING CYCLES OVER $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$

Let S denote a base scheme. In [25], the second author introduced a formalism similar to that of inertia invariant vanishing cycles. In order to set up a complete theory, the language of stacks is needed. Indeed, we will define a sheaf of (tame) vanishing cycles for a scheme X together with a global section s of a line bundle \mathcal{L} . The datum of the triplet (X, \mathcal{L}, s) is equivalent to that of a morphism $X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$. In fact, such a morphism is the datum of a diagram

$$(4.0.1) \quad X \leftarrow P \rightarrow \mathbb{A}_S^1,$$

where $P \rightarrow X$ is a $\mathbb{G}_{m,S}$ -torsor and $P \rightarrow \mathbb{A}_S^1$ a $\mathbb{G}_{m,S}$ -equivariant morphism.

The main idea is then to let $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ play the role of a disk. Then, $B\mathbb{G}_{m,S}$ will play the role of the center of the disk, while $S \simeq [\mathbb{A}_S^1 - \{0\}/\mathbb{G}_{m,S}] \simeq [\mathbb{A}_S^1/\mathbb{G}_{m,S}] - B\mathbb{G}_{m,S}$ that of the punctured disk. In the introduction of [25] there is a mental map with an analogy between the different disks in our, the topological and the geometric/arithmetical setting.

Let $X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ be an X -point of $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, corresponding to (X, \mathcal{L}, s) , where \mathcal{L} is a line bundle on X and $s \in H^0(X, \mathcal{L})$. We will consider the diagram

$$(4.0.2) \quad \begin{array}{ccccc} X_0 & \longrightarrow & X & \longleftarrow & U_X \\ \downarrow & & \downarrow & & \downarrow \\ B\mathbb{G}_{m,S} & \longrightarrow & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \longleftarrow & S, \end{array}$$

where both squares are Cartesian.

Remark 4.1. Notice that $X_0 \simeq V(s)$ is the zero locus of s , while $U_X \simeq X - X_0 \simeq X - V(s)$ is its open complementary.

Moreover, let $\mathbb{A}_S^1 \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ be the canonical atlas of $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ (i.e., the \mathbb{A}_S^1 -point of $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ corresponding to $(\mathbb{A}_S^1, \mathcal{O}_S[t], t)$). By pulling back the diagram above along this smooth morphism we get

$$(4.1.1) \quad \begin{array}{ccccc} P_{X_0} & \longrightarrow & P_X & \longleftarrow & P_{U_X} \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & \mathbb{A}_S^1 & \longleftarrow & \mathbb{G}_{m,S}. \end{array}$$

All morphisms in this diagram are $\mathbb{G}_{m,S}$ -equivariant. Also, the canonical morphism $P_X \rightarrow X$ (resp. $P_{X_0} \rightarrow X_0$) is the $\mathbb{G}_{m,S}$ -torsor $(\mathbb{V}(\mathcal{L}) - X) \rightarrow X$ (resp. $(\mathbb{V}(\mathcal{L}_{|X_0}) - X_0) \rightarrow X_0$).

Notation 4.2. Let \mathbb{N}_S denote the set of natural numbers that are invertible in \mathcal{O}_S . We will consider it as a filtered set, where $n \leq m$ if and only if n divides m .

Construction 4.3. Let $n \in \mathbb{N}_S$. Elevation to the n^{th} -power of the parameter of \mathbb{A}_S^1 induces a morphism of Artin stacks

$$(4.3.1) \quad \Theta^{(n)} : [\mathbb{A}_S^1/\mathbb{G}_{m,S}] \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}].$$

More precisely, it is the morphism of quotient stacks induced by the commutative square

$$(4.3.2) \quad \begin{array}{ccc} \mathbb{G}_{m,S} \times \mathbb{A}_S^1 \rightarrow \mathbb{A}_S^1 & (u, x) \mapsto & u \cdot x \\ \downarrow & \downarrow & \downarrow \\ \mathbb{G}_{m,S} \times \mathbb{A}_S^1 \rightarrow \mathbb{A}_S^1 & (u^n, x^n) \mapsto & u^n \cdot x^n = (u \cdot x)^n. \end{array}$$

As functor of points, for an S -scheme X , the induced functor $[\mathbb{A}_S^1/\mathbb{G}_{m,S}](X) \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}](X)$ sends an object (\mathcal{L}, s) to $(\mathcal{L}^{\otimes n}, s^{\otimes n})$.

Remark 4.4. These morphisms were first introduced (independently) in [7] and [1].

Remark 4.5. For every $n, m \in \mathbb{N}_S$, there are canonical equivalences of morphisms of Artin stacks

$$(4.5.1) \quad \Theta^{(n)} \circ \Theta^{(m)} \simeq \Theta^{(nm)} \simeq \Theta^{(m)} \circ \Theta^{(n)}.$$

Definition 4.6. Let $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ be a morphism of Artin stacks. For every $n \in \mathbb{N}_S$ we define

$$(4.6.1) \quad X^{(n)} := X \times_{p, [\mathbb{A}_S^1/\mathbb{G}_{m,S}], \Theta^{(n)}} [\mathbb{A}_S^1/\mathbb{G}_{m,S}],$$

$$(4.6.2) \quad X_0^{(n)} := X_0 \times_X X^{(n)}.$$

For every $n, m \in \mathbb{N}_S$ we dispose of a diagram of Artin stacks

$$(4.6.3) \quad \begin{array}{ccccc} X_0^{(nm)} & \xrightarrow{i_X^{(nm)}} & X^{(nm)} & \xleftarrow{j_X^{(nm)}} & U_X \\ \downarrow u_{X_0, (n)}^{(nm)} & & \downarrow u_{X, (n)}^{(nm)} & & \downarrow id_{U_X} \\ X_0^{(n)} & \xrightarrow{i_X^{(n)}} & X^{(n)} & \xleftarrow{j_X^{(n)}} & U_X. \end{array}$$

where all squares are Cartesian.

It follows immediately from the functoriality of fibre products that we dispose of diagrams of Artin stacks

$$(4.6.4) \quad X_0^{(\bullet)}, X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Art}_S.$$

Moreover, if we consider U_X as a constant diagram $\mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Art}_S$, we have morphisms

$$(4.6.5) \quad i_X^{(\bullet)} : X_0^{(\bullet)} \rightarrow X^{(\bullet)} \leftarrow U_X : j_X^{(\bullet)}.$$

Notation 4.7. Let $n \in \mathbb{N}_S$ and let Λ be a ring of coefficients as in section §1. Let μ_n denote the group $\mathcal{O}_S[t]/(t^n - 1)$. We will adopt the following notation:

$$(4.7.1) \quad \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_n} := \mathcal{D}_{\text{ét}}(X^{(n)}; \Lambda),$$

$$(4.7.2) \quad \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n} := \mathcal{D}_{\text{ét}}(X_0^{(n)}; \Lambda).$$

The reason why we introduced the above notation is that $X^{(n)}$ (resp. $X_0^{(n)}$) plays the role of $[X/\mu_n]$ (resp. $[X_0/\mu_n]$)². Therefore, we are justified to think of complexes on $X^{(n)}$ (resp. $X_0^{(n)}$) as complexes on X (resp. X_0) endowed with an action of μ_n . From this perspective, the adjunctions $((u_{X_0, (n)}^{(nm)})^*, (u_{X_0, (n)}^{(nm)})_*)$, $((u_{X, (n)}^{(nm)})^*, (u_{X, (n)}^{(nm)})_*)$ identify with

$$(4.7.3) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n} \rightleftarrows \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_{nm}} : (-)^{\mu_m},$$

$$(4.7.4) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_n} \rightleftarrows \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_{nm}} : (-)^{\mu_m}.$$

Let

$$(4.7.5) \quad \begin{array}{c} \mathbf{Art}_S \rightarrow \mathbf{Pr}_{\text{stb}}^{\text{R}} \\ (f : \mathfrak{Y} \rightarrow \mathfrak{X}) \mapsto (f_* : \mathcal{D}_{\text{ét}}(\mathfrak{Y}; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathfrak{X}; \Lambda)) \end{array}$$

and consider the diagram of presentable stable ∞ -categories

$$(4.7.6) \quad (4.7.5) \circ X_0^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Pr}_{\text{stb}}^{\text{R}}.$$

Definition 4.8. We define the ∞ -category $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ as the limit of the diagram above:

$$(4.8.1) \quad \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} := \varprojlim_{\mathbb{N}_S} \mathcal{D}_{\text{ét}}(X_0^{(\bullet)}; \Lambda) \in \mathbf{Pr}_{\text{stb}}^{\text{R}}.$$

Similarly, we define the ∞ -category $\mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_\infty}$ as the limit of the diagram

$$(4.8.2) \quad (4.7.5) \circ X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Pr}_{\text{stb}}^{\text{R}}.$$

Objects in these presentable stable ∞ -category should be thought as complexes on X_0 (resp. X) endowed with a *continuous* action of $\mu_\infty := \varprojlim \mu_n$.

Remark 4.9. Notice that $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ is also a limit in $\widehat{\mathcal{C}at}_\infty$, the big ∞ -category of ∞ -categories. See [20, Theorem 5.5.3.18] and [21, Theorem 1.1.4.4].

²Notice that, however, $X_0^{(n)}$ is a nilthickening of $[X_0/\mu_n]$

For every $n \in \mathbb{N}_S$ there is a short exact sequence of profinite groups

$$(4.9.1) \quad 1 \rightarrow n \cdot \mu_\infty \rightarrow \mu_\infty \rightarrow \mu_n \rightarrow 1.$$

The canonical projection $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n}$ is the right adjoint in the adjunction

$$(4.9.2) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n} \rightleftarrows \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} : (-)^{n \cdot \mu_\infty}$$

that corresponds to taking fixed points with respect to the action of the subgroup $n \cdot \mu_\infty \subseteq \mu_\infty$.

In particular, we dispose of the adjunction

$$(4.9.3) \quad \text{triv} : \mathcal{D}_{\text{ét}}(X_0; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} : (-)^{\mu_\infty}.$$

Definition 4.10. We define $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ as the recollement determined by the adjunction 4.9.3.

Lemma 4.11. $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ is a presentable stable ∞ -category.

Proof. $\mathcal{D}_{\text{ét}}(X_0; \Lambda)$ and $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ are presentable and by [20, Proposition 5.4.7.7] $(-)^{\mu_\infty}$ is an accessible functor. By [20, Corollary 5.4.7.17], $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ is accessible. As it also admits small colimits, $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ is presentable.

It is stable by [21, Proposition A.8.17].

Alternatively, by Proposition 2.12 one sees that $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ is a fiber product in $\mathbf{Pr}_{\text{stb}}^{\mathbf{R}}$, which is a complete (big) ∞ -category ([20, Theorem 5.5.3.18]). \square

Tame nearby cycles over $[\mathbb{A}_S^1/\mathbb{G}_{\mathbf{m},S}]$. We will define a tame nearby cycles functor in this $\mathbb{G}_{\mathbf{m},-}$ equivariant setting following the lines of 3.14.10.

Construction 4.12. Consider the diagram

$$(4.12.1) \quad j_X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \text{Fun}(\Delta^1, \mathbf{Art}_S).$$

We define the ∞ -functor

$$(4.12.2) \quad (\bar{j}_X)_* : \mathcal{D}_{\text{ét}}(U_X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_\infty},$$

as the limit of the diagram

$$(4.12.3) \quad (4.7.5) \circ j_X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\mathbf{R}}).$$

The next step will be constructing the ∞ -functor

$$(4.12.4) \quad \bar{i}_X^* : \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}.$$

For the same reason we explained in Remark 3.11, we will not define it as the left adjoint to the limit of the diagram

$$(4.12.5) \quad (4.7.5) \circ i_X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\mathbf{R}}),$$

but rather we will follow the same path of Construction 3.12.

Construction 4.13. Consider the diagram $X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Art}_S$. Also consider the closed embedding $X_0 \hookrightarrow X$. Then we can use the construction of Appendix A and get a diagram

$$(4.13.1) \quad \mathbb{N}_S^{\text{op}} \rightarrow \text{Fun}(\Delta^1, \text{Corr}(N(\mathbf{Art}_S))_{F, \text{all}}).$$

We can compose it with the ∞ -functor 1.0.27 and get a diagram

$$(4.13.2) \quad \mathbb{N}_S^{\text{op}} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\mathbf{L}}).$$

Finally, using the equivalences $\mathbf{Pr}_{\text{stb}}^L \simeq (\mathbf{Pr}_{\text{stb}}^R)^{op}$ and $\Delta^1 \simeq (\Delta^1)^{op}$ we get another diagram

$$(4.13.3) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R)$$

that expresses the functoriality of the equivalences

$$(-)^{m \cdot \mu_\infty} \circ (i_X^{(nm)})^* \simeq (i_X^{(n)})^* \circ (-)^{m \cdot \mu_\infty}.$$

We define 4.12.4 as its limit.

Remark 4.14. One can construct the ∞ -functor 4.12.4 also using the formalism of [12]. Consider the ∞ -functor 4.7.5. Moreover, let pDM denote the class of proper morphisms representable by DM stacks in \mathbf{Art}_S . By the results reminded in Section 1, every morphism in pDM satisfies the left Beck-Chevalley condition. Then, by [12, Theorem 3.2.2, Chapter 7] we have the functor

$$(4.14.1) \quad \text{Corr}(\mathbf{Art}_S)_{all, pDM}^{pDM} \rightarrow (\infty, 2)\mathbf{Pr}_{\text{stb}}^R.$$

Similarly to what we did in Construction 3.12, we shall restrict it to

$$(4.14.2) \quad \text{Corr}(\mathbf{Art}_S)_{all, pDM}^{pceq} \rightarrow \mathbf{Pr}_{\text{stb}}^R,$$

where $pceq \subseteq pDM$ denotes the class of proper morphism representable by DM stacks such that the induced pushforward is an equivalence. Just as in Construction 3.12, at the level of objects this ∞ -functor sends an Artin stack \mathfrak{X} to the ∞ -category $\mathcal{D}_{\text{ét}}(\mathfrak{X}; \Lambda)$ and it sends a correspondence

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow g & & \\ \mathfrak{Z} & & \end{array}$$

to the ∞ -functor $g_* \circ f^! : \mathcal{D}_{\text{ét}}(\mathfrak{Y}; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathfrak{Z}; \Lambda)$.

Consider the composition ∞ -functors $(\mathbf{Art}_S)_{pDM} \rightarrow (\mathbf{Art}_S)_{all} \rightarrow \text{Corr}(\mathbf{Art}_S)_{all, pDM}^{pDM}$ with diagram $i_X^{(\bullet)} : X'_{\bullet} \rightarrow X^{(\bullet)}$, where $X'_{\bullet} = X_{\bullet} \times_{p_{\bullet}, \text{BG}_{m,S}, \Theta_{\bullet}^{(n)}} \text{BG}_{m,S}$. This way, we get a diagram

$$(4.14.3) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \text{Corr}(\mathbf{Art}_S)_{all, pDM}^{pDM}).$$

For n, m in \mathbb{N}_S , the associated morphism of correspondences of Artin stacks is

$$(4.14.4) \quad \begin{array}{ccc} X'_{\bullet}^{(nm)} & \xrightarrow{u_{X_{\bullet}, (n)}^{(nm)}} & X'_{\bullet}^{(n)} \\ \searrow^{i_X^{(nm)}} & \downarrow & \downarrow i_X^{(n)} \\ X'_{\bullet}^{(n)} \times_{X^{(n)}} X^{(nm)} & \xrightarrow{u_{X, (n)}^{(nm)}} & X'_{\bullet}^{(n)} \\ \downarrow & & \downarrow \\ X^{(nm)} & \xrightarrow{u_{X, (n)}^{(nm)}} & X^{(n)} \end{array}$$

The morphisms $X'_{\bullet}^{(nm)} \rightarrow X'_{\bullet}^{(n)} \times_{X^{(n)}} X^{(nm)}$ are nilthickenings and therefore they belong to pDM . Thus, we have an induced diagram

$$(4.14.5) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \text{Corr}(\mathbf{Art}_S)_{all, pDM}^{pceq}).$$

By composing it with 4.14.2, we get a diagram

$$(4.14.6) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R),$$

which provides us with the functoriality of the equivalences

$$(i'_X)^{(nm)} \circ (u_{X,(n)}^{(nm)})^! \simeq (u_{X_0,(n)}^{(nm)})^! \circ (i'_X)^{(n)}.$$

Passing to left adjoints, i.e. using the equivalences $\mathbf{Pr}_{\text{stb}}^R \simeq (\mathbf{Pr}_{\text{stb}}^L)^{op}$ and $\Delta^1 \simeq (\Delta^1)^{op}$ as in Construction 3.12, we get a diagram

$$(4.14.7) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^L)$$

that expresses the functoriality of the equivalences

$$(-)^{m \cdot \mu_\infty} \circ (i'_X)^{(nm)} \circ (-)^* \simeq (i'_X)^{(n)} \circ (-)^* \circ (-)^{m \cdot \mu_\infty}.$$

We define 4.12.4 as the limit of this diagram.

The preservation of (small) limits of both inclusions $\mathbf{Pr}_{\text{stb}}^L \subseteq \widehat{\text{Cat}}_\infty$ and $\mathbf{Pr}_{\text{stb}}^R \subseteq \widehat{\text{Cat}}_\infty$ guarantees that the source (resp. target) of this ∞ -functor is indeed $\mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_\infty}$ (resp. $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$).

Definition 4.15. We define *tame nearby cycles over* $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ as the ∞ -functor

$$(4.15.1) \quad \Psi_{\eta,p}^t := \bar{i}_X^* \circ (\bar{j}_X)_* : \mathcal{D}_{\text{ét}}(U_X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}.$$

For $\mathcal{F} \in \mathcal{D}_{\text{ét}}(U_X; \Lambda)$, we will refer to $\Psi_{\eta,p}^t(\mathcal{F})$ as the *sheaf of tame nearby cycles with coefficients in* \mathcal{F} .

Tame vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$.

Remark 4.16. There is an equivalence of ∞ -functors

$$(4.16.1) \quad (i_X^{(1)})^* \circ (j_X^{(1)})_* \simeq (-)^{\mu_\infty} \circ \Psi_{\eta,p}^t : \mathcal{D}_{\text{ét}}(U_X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

Indeed, by construction we have that

$$(4.16.2) \quad (-)^{\mu_\infty} \circ \bar{i}_X^* \simeq (i_X^{(1)})^* \circ (-)^{\mu_\infty}, \quad (-)^{\mu_\infty} \circ (\bar{j}_X)_* \simeq (j_X^{(1)})_*.$$

The previous remark provides us with a natural transformation

$$(4.16.3) \quad i_X^* = (i_X^{(1)})^* \rightarrow (-)^{\mu_\infty} \circ \Psi_{\eta,p}^t \circ (j_X^{(1)})_* \simeq (i_X^{(1)})^* \circ (j_X^{(1)})_* \circ (j_X^{(1)})^* : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda),$$

induced by the unit of the adjunction $((j_X^{(1)})^*, (j_X^{(1)})_*)$.

Definition 4.17. Define the ∞ -functor of *total tame vanishing cycles over* $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$

$$(4.17.1) \quad \Psi_p^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$$

to be the functor obtained by applying Corollary 2.7 to $(i_X^*, \Psi_{\eta,p}^t \circ (j_X^{(1)})_*, i_X^* \rightarrow (-)^{\mu_\infty} \circ \Psi_{\eta,p}^t \circ (j_X^{(1)})_*)$.

For $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X; \Lambda)$, we refer to $\Psi_p^t(\mathcal{F})$ as the *sheaf of total tame vanishing cycles with coefficients in* \mathcal{F} .

Definition 4.18. We define *tame vanishing cycles over* $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ as the ∞ -functor

$$(4.18.1) \quad \Phi_p^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty},$$

obtained as the following composition

$$(4.18.2) \quad \mathcal{D}_{\text{ét}}(X; \Lambda) \xrightarrow{\Psi_p^t} \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \xrightarrow{sp} \text{Fun}(\Delta^1, \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}) \xrightarrow{cofib} \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}.$$

For $\mathcal{F} \in \mathcal{D}_{\text{ét}}(X; \Lambda)$, we refer to $\Phi_p^t(\mathcal{F})$ as the *sheaf of tame vanishing cycles with coefficients in* \mathcal{F} .

5. FUNCTORIAL BEHAVIOUR OF TAME NEARBY AND VANISHING CYCLES OVER $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$

Let $f : Y \rightarrow X$ be a morphism over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, i.e. assume that

$$(5.0.1) \quad \begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow q & \swarrow p \\ & & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] \end{array}$$

commutes.

Remark 5.1. Assume that X and Y are S -schemes. Explicitly, this means that if p is determined by $(X \rightarrow S, \mathcal{L}, s)$ and q by $(Y \rightarrow S, \mathcal{M}, t)$, then f is an S -morphism and we are given an isomorphism $\alpha : \mathcal{M} \simeq f^* \mathcal{L}$ such that t corresponds to $f^*(s)$ under α .

Then we have the following commutative diagrams

$$(5.1.1) \quad \begin{array}{ccccc} Y_0^{(n)} & \xrightarrow{i_Y^{(n)}} & Y^{(n)} & \xleftarrow{j_Y^{(n)}} & U_Y \\ \downarrow f_0^{(n)} & & \downarrow f^{(n)} & & \downarrow f_U \\ X_0^{(n)} & \xrightarrow{i_X^{(n)}} & X^{(n)} & \xleftarrow{j_X^{(n)}} & U_X. \end{array}$$

We will now define an adjunction

$$(5.1.2) \quad f_{\mathcal{Y}^t}^* : \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) : f_{\mathcal{Y}^t,*}$$

using the results we discussed in Section 2.

Consider the adjunction

$$(5.1.3) \quad f_0^* = (f_0^{(1)})^* : \mathcal{D}_{\text{ét}}(X_0; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(Y_0; \Lambda) : (f_0^{(1)})_* = (f_0)_*$$

induced by pullback/pushforward along $f_0 = f_0^{(1)} : Y_0 \rightarrow X_0$.

Clearly, the maps $f_0^{(n)} : Y_0^{(n)} \rightarrow X_0^{(n)}$ arrange in a morphism of diagrams of Artin stacks

$$(5.1.4) \quad f_0^{(\bullet)} : Y_0^{(\bullet)} \rightarrow X_0^{(\bullet)} : \mathbb{N}_S^{op} \rightarrow \mathbf{Art}_S.$$

In particular, we have adjunctions

$$(5.1.5) \quad (f_0^{(n)})^* : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n} \rightleftarrows \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_n} : (f_0^{(n)})_*.$$

Construction 5.2. We will define an adjunction

$$(5.2.1) \quad (\bar{f}_0)^* : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \rightleftarrows \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty} : (\bar{f}_0)_*.$$

In order to do so, consider the diagram

$$(5.2.2) \quad X_0^{(\bullet)} : \mathbb{N}_S^{op} \rightarrow \mathbf{Art}_S.$$

By considering the morphism $Y_0 \rightarrow X_0$ and applying the construction in Appendix A, we get a diagram

$$(5.2.3) \quad \mathbb{N}_S^{op} \times \Delta^1 \rightarrow \text{Corr}(N(\mathbf{Art}_S))_{F,all}.$$

If we then compose it with the ∞ -functor 1.0.27, we obtain a diagram

$$(5.2.4) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^L)$$

which gives us homotopy coherent equivalences

$$(5.2.5) \quad (-)^{\mu_m} \circ (f_0^{(nm)})^* \simeq (f_0^n)^* \circ (-)^{\mu_m}.$$

We define $(\bar{f}_0)^* : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty}$ as the limit of this diagram and $(\bar{f}_0)_* : \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ as its right adjoint.

Then, the adjunction (5.1.2) is obtained by applying Proposition 2.17 to (5.1.3) and (5.2.1).

Similarly, if f is a separated morphism locally of finite type, one can define an adjunction

$$(5.2.6) \quad f_{\mathcal{Y}^t, !} : \mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) : f_{\mathcal{Y}^t}^!$$

As above, one considers two adjunctions

$$(5.2.7) \quad (f_0)_! = (f_0^{(1)})_! : \mathcal{D}_{\text{ét}}(Y_0; \Lambda) \rightleftarrows \mathcal{D}_{\text{ét}}(X_0; \Lambda) : (f_0^{(1)})^! = (f_0)^!,$$

$$(5.2.8) \quad (\bar{f}_0)_! : \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty} \rightleftarrows \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} : (\bar{f}_0)^!.$$

Construction 5.3. We consider the same diagram

$$(5.3.1) \quad \mathbb{N}_S \times \Delta^1 \rightarrow \text{Corr}(N(\mathbf{Art}_S))_{F, \text{all}}$$

as above, i.e. the one obtained by applying the construction in Appendix A to $X_0^{(\bullet)}$ and $Y_0 \rightarrow X_0$. We composite it with 1.0.27 and then we pass to right adjoints, thus obtaining a diagram

$$(5.3.2) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R)$$

which provides us with homotopy coherent equivalences

$$(5.3.3) \quad (-)^{\mu_m} \circ (f_0^{(nm)})^! \simeq (f_0^n)^! \circ (-)^{\mu_m}.$$

We define $(\bar{f}_0)^! : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \rightleftarrows \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty}$ as the limit of this diagram and $(\bar{f}_0)_! : \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty} \rightleftarrows \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ as its left adjoint.

Then, the adjunction $(f_{\mathcal{Y}^t, !}, f_{\mathcal{Y}^t}^!)$ is defined by applying Proposition 2.17.

Tame vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{\mathbf{m}, S}]$ and $*$ -pullbacks. The aim of the present subsection is to define a morphism of ∞ -functors

$$(5.3.4) \quad f_{\mathcal{Y}^t}^* \circ \Psi_p^t \rightarrow \Psi_q^t \circ f^* : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda).$$

By composing both $f_{\mathcal{Y}^t}^* \circ \Psi_p^t$ and $\Psi_q^t \circ f^*$ with $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)$ we find the ∞ -functors $f_0^* \circ i_X^*$ and $i_Y^* \circ f^*$, which are clearly equivalent.

On the other hand, if we compose $f_{\mathcal{Y}^t}^* \circ \Psi_p^t$ and $\Psi_q^t \circ f^*$ with $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty}$ we obtain the ∞ -functors $\bar{f}_0^* \circ \Psi_{\eta, p}^t \circ j_X^*$ and $\Psi_{\eta, q}^t \circ f^* \circ j_Y^*$ respectively.

Construction 5.4. By repeating Construction 5.2 replacing diagram 5.1.4 with

$$(5.4.1) \quad f^{(\bullet)} : Y^{(\bullet)} \rightarrow X^{(\bullet)}$$

we get the ∞ -functor

$$(5.4.2) \quad \bar{f}^* : \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(Y; \Lambda)^{\mu_\infty}$$

and its right adjoint

$$(5.4.3) \quad \bar{f}_* : \mathcal{D}_{\text{ét}}(Y; \Lambda)^{\mu\infty} \rightarrow \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu\infty}.$$

Remark 5.5. One can verify that $\bar{f}_* : \mathcal{D}_{\text{ét}}(Y; \Lambda)^{\mu\infty} \rightarrow \mathcal{D}_{\text{ét}}(X; \Lambda)^{\mu\infty}$ can be obtained as the limit of the diagram

$$(5.5.1) \quad \mathbb{N}_S^{\text{op}} \xrightarrow{f^{(\bullet)}} \text{Fun}(\Delta^1, \mathbf{Art}_S) \xrightarrow{(4.7.5)} \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}}).$$

Lemma 5.6. (1) *There is an equivalence of ∞ -functors $\bar{f}_0^* \circ \bar{i}_X^* \simeq \bar{i}_Y^* \circ \bar{f}^*$.*

(2) *There is a natural transformation $\alpha_f : \bar{f}^* \circ (\bar{j}_X)_* \rightarrow (\bar{j}_Y)_* \circ f_U^*$.*

(3) *If f is smooth, then α_f is an equivalence.*

Proof. (1) Notice that $\bar{f}_0^* \circ \bar{i}_X^*$ can be obtained by applying Construction 5.2 to the diagram of Artin stacks

$$(5.6.1) \quad f_0^{(\bullet)} \circ i_X^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Art}_S.$$

Similarly, $\bar{i}_Y^* \circ \bar{f}^*$ can be obtained by applying Construction 5.2 to the diagram of Artin stacks

$$(5.6.2) \quad i_Y^{(\bullet)} \circ f^{(\bullet)} : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Art}_S.$$

As these diagrams are clearly equivalent, the claim follows.

(2) By adjunction, α_f corresponds to a natural transformation $(\bar{j}_X)_* \rightarrow \bar{f}_* \circ (\bar{j}_Y)_* \circ f_U^*$. By Remark 5.5, we see that the ∞ -functor $\bar{f}_* \circ (\bar{j}_Y)_*$ can be obtained as the limit of the diagram

$$(5.6.3) \quad \mathbb{N}_S^{\text{op}} \xrightarrow{f^{(\bullet)} \circ j_Y^{(\bullet)}} \text{Fun}(\Delta^1, \mathbf{Art}_S) \xrightarrow{(4.7.5)} \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}}).$$

As the diagram of Artin stacks $f^{(\bullet)} \circ j_Y^{(\bullet)}$ is equivalent to $j_X^{(\bullet)} \circ f_U$, we see that $\bar{f}_* \circ (\bar{j}_Y)_*$ is equivalent to the limit of the diagram

$$(5.6.4) \quad \mathbb{N}_S^{\text{op}} \xrightarrow{j_X^{(\bullet)} \circ f_U} \text{Fun}(\Delta^1, \mathbf{Art}_S) \xrightarrow{(4.7.5)} \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}}),$$

i.e. to $(\bar{j}_X)_* \circ (f_U)_*$. Then $(\bar{j}_X)_* \rightarrow \bar{f}_* \circ (\bar{j}_Y)_* \circ f_U^*$ is the morphism induced by the unit of the adjunction $(f_U^*, (f_U)_*)$.

(3) It suffices to show that, for $\mathcal{F} \in \mathcal{D}_{\text{ét}}(U_Y; \Lambda)$, the morphism $\alpha_f(\mathcal{F}) : \bar{f}^* \circ (\bar{j}_Y)_*(\mathcal{F}) \rightarrow (\bar{j}_X)_* \circ f_U^*(\mathcal{F})$ is an equivalence. By construction,

$$(5.6.5) \quad \bar{f}^* \circ (\bar{j}_Y)_*(\mathcal{F}) \simeq \varinjlim_{n \in \mathbb{N}_S} (f^{(n)})^* \circ (j_Y^{(n)})_*(\mathcal{F}), \quad (\bar{j}_X)_* \circ f_U^*(\mathcal{F}) \simeq \varinjlim_{n \in \mathbb{N}_S} (j_X^{(n)})^* \circ f_U^*(\mathcal{F}).$$

Under these equivalences, $\alpha_f(\mathcal{F})$ corresponds to the colimit of the base change morphisms $(f^{(n)})^* \circ (j_Y^{(n)})_*(\mathcal{F}) \rightarrow (j_X^{(n)})^* \circ f_U^*(\mathcal{F})$, which are equivalences by smooth base change. \square

We define a natural transformation

$$(5.6.6) \quad \bar{f}_0^* \circ \Psi_{\eta,p}^t \circ j_X^* \rightarrow \Psi_{\eta,q}^t \circ j_Y^* \circ f^*$$

as the morphism

$$(5.6.7) \quad \bar{f}_0^* \circ \bar{i}_X^* \circ (\bar{j}_X)_* \circ j_X^* \simeq \bar{i}_Y^* \circ \bar{f}^* \circ (\bar{j}_X)_* \circ j_X^* \xrightarrow{\alpha_f} \bar{i}_Y^* \circ (\bar{j}_Y)_* \circ f_U^* \circ j_X^* \simeq \bar{i}_Y^* \circ (\bar{j}_Y)_* \circ j_X^* \circ f^*.$$

Lemma 5.7. *Applying $(-)^{\mu_\infty}$ to (5.6.6) we obtain (up to homotopy) the morphism*

$$(5.7.1) \quad f_0^* \circ (i_X^{(1)})^* \circ (j_X^{(1)})_* \circ (j_X^{(1)})^* \rightarrow (i_Y^{(1)})^* \circ (j_Y^{(1)})_* \circ (j_Y^{(1)})^* \circ f^*.$$

Proof. By construction, we have that

$$(5.7.2) \quad (-)^{\mu_\infty} \circ \bar{f}_0^* \simeq f_0^* \circ (-)^{\mu_\infty}.$$

Then, using also Remark 4.16, we get that

$$(5.7.3) \quad (\bar{f}_0^* \circ \Psi_{\eta,p}^t \circ j_X^* \rightarrow \Psi_{\eta,q}^t \circ j_Y^* \circ f^*)^{\mu_\infty} \simeq f_0^* \circ (i_X^{(1)})^* \circ (j_X^{(1)})_* \circ (j_X^{(1)})^* \rightarrow (i_Y^{(1)})^* \circ (j_Y^{(1)})_* \circ (j_Y^{(1)})^* \circ f^*,$$

where the morphism on the right is induced by the base change morphism $f^* \circ (j_X^{(1)})_* \rightarrow (j_Y^{(1)})_* \circ f^*$. \square

It is then clear that we have a square $\sigma : \Delta^1 \times \Delta^1 \rightarrow \text{Fun}(\mathcal{D}_{\text{ét}}(X; \Lambda), \mathcal{D}_{\text{ét}}(Y_0; \Lambda))$

$$(5.7.4) \quad \begin{array}{ccc} f_0^* \circ i_X^* & \longrightarrow & f_0^* \circ (i_X^{(1)})^* \circ (j_X^{(1)})_* \circ (j_X^{(1)})^* \\ \downarrow \sim & & \downarrow \\ i_Y^* \circ f^* & \longrightarrow & (i_Y^{(1)})^* \circ (j_Y^{(1)})_* \circ (j_Y^{(1)})^* \circ f^*. \end{array}$$

and therefore the data $(f_0^* \circ i_X^* \simeq i_Y^* \circ f^*, \tilde{f}_0^* \circ \Psi_{\eta,p}^t \circ j_X^* \rightarrow \Psi_{\eta,q}^t \circ j_Y^* \circ f^*, \sigma)$ define the morphism (5.3.4).

Proposition 5.8. *If $f : Y \rightarrow X$ is a smooth morphism, then (5.3.4) is an equivalence.*

Proof. As the ∞ -functors $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)$ and $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0)^{\mu_\infty}$ detect equivalences, it suffices to show that (5.6.6) is an equivalence. But this is a consequence of Lemma 5.6. \square

Tame vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ and $*$ -pushforward. The aim of the present subsection is to define a morphism of ∞ -functors

$$(5.8.1) \quad \Psi_p^t \circ f_* \rightarrow (f_{\mathcal{Y}^t})_* \circ \Psi_q^t : \mathcal{D}_{\text{ét}}(Y; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda).$$

Composing $\Psi_p^t \circ f_*$ and $(f_{\mathcal{Y}^t})_* \circ \Psi_q^t$ with $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)$ we obtain the ∞ -functors $i_X^* \circ f_*$ and $(f_0)_* \circ i_Y^*$ respectively. Then, we clearly have a morphism

$$(5.8.2) \quad i_X^* \circ f_* \rightarrow (f_0)_* \circ i_Y^* : \mathcal{D}_{\text{ét}}(Y; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

If we compose them with $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty}$ instead, we get the ∞ -functors $\Psi_{\eta,p}^t \circ j_X^* \circ f_*$ and $(\tilde{f}_0)_* \circ \Psi_{\eta,q}^t \circ j_Y^*$ respectively.

Lemma 5.9. (1) *There is an equivalence of ∞ -functors $(\bar{i}_X)_* \circ (\bar{f}_0)_* \simeq \bar{f}_* \circ (\bar{i}_Y)_*$.*

(2) *There is a natural transformation $\beta_f : \bar{i}_X^* \circ \bar{f}_* \rightarrow (\bar{f}_0)_* \circ \bar{i}_Y^*$.*

(3) *If f is a proper morphism representable by DM stacks, then β_f is an equivalence.*

Proof. (1) This follows from the observation that the diagrams $i_X^{(\bullet)} \circ f^{(\bullet)} : \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Art}_S)$ and $f_0^{(\bullet)} \circ i_Y^{(\bullet)} : \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \mathbf{Art}_S)$ are equivalent and that $(\bar{i}_X)_* \circ (\bar{f}_0)_*$ (resp. $\bar{f}_* \circ (\bar{i}_Y)_*$) can be obtained as the limit of $\mathbb{N}_S^{op} \xrightarrow{i_X^{(\bullet)} \circ f^{(\bullet)}} \text{Fun}(\Delta^1, \mathbf{Art}_S) \xrightarrow{4.7.5} \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R)$ (resp. $\mathbb{N}_S^{op} \xrightarrow{f_0^{(\bullet)} \circ i_Y^{(\bullet)}} \text{Fun}(\Delta^1, \mathbf{Art}_S) \xrightarrow{4.7.5} \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R)$).

- (2) The natural transformation β_f corresponds by adjunction to a morphism $\bar{f}_* \rightarrow (\bar{i}_X)_* \circ (\bar{f}_0)_* \circ \bar{i}_Y^* \simeq \bar{f}_* \circ (\bar{i}_Y)_* \circ \bar{i}_X^*$, which is induced by the unit of the adjunction $(\bar{i}_Y^*, (\bar{i}_Y)_*)$.
- (3) It suffices to show that, for $\mathcal{F} \in \mathcal{D}_{\text{ét}}(Y; \Lambda)^{\mu_\infty}$, the morphism $\beta_f(\mathcal{F}) : \bar{i}_X^* \circ \bar{f}_*(\mathcal{F}) \rightarrow (\bar{f}_0)_* \circ \bar{i}_Y^*(\mathcal{F})$ is an equivalence. By construction,

$$(5.9.1) \quad \bar{i}_X^* \circ \bar{f}_*(\mathcal{F})(\mathcal{F}) \simeq \varinjlim_{n \in \mathbb{N}_S} (i_X^{(n)})^* \circ (f^{(n)})_*(\mathcal{F}), \quad (\bar{f}_0)_* \circ \bar{i}_Y^*(\mathcal{F}) \simeq \varinjlim_{n \in \mathbb{N}_S} (f_0^{(n)})_* \circ (i_Y^{(n)})^*(\mathcal{F}).$$

Under these equivalences, $\beta_f(\mathcal{F})$ corresponds to the colimit of the base change morphisms $(i_X^{(n)})^* \circ (f^{(n)})_*(\mathcal{F}) \rightarrow (f_0^{(n)})_* \circ (i_Y^{(n)})^*(\mathcal{F})$, which are equivalences by proper base change. \square

We define

$$(5.9.2) \quad \Psi_{\eta,p}^t \circ j_X^* \circ f_* \rightarrow (\bar{f}_0)_* \circ \Psi_{\eta,q}^t \circ j_Y^*$$

as the natural transformation

$$(5.9.3) \quad \bar{i}_X^* \circ (\bar{j}_X)_* \circ j_X^* \circ f_* \rightarrow \bar{i}_X^* \circ (\bar{j}_X)_* \circ (f_U)_* \circ j_Y^* \simeq \bar{i}_X^* \circ \bar{f}_* \circ (\bar{j}_Y)_* \circ j_Y^* \xrightarrow{\beta_f} (\bar{f}_0)_* \circ \bar{i}_Y^* \circ (\bar{j}_Y)_* \circ j_Y^*.$$

Lemma 5.10. *The square $\tau : \partial(\Delta^1 \times \Delta^1) \rightarrow \text{Fun}(\mathcal{D}_{\text{ét}}(Y; \Lambda), \mathcal{D}_{\text{ét}}(X_0; \Lambda))$*

$$(5.10.1) \quad \begin{array}{ccc} (i_X^{(1)})^* \circ f_* & \longrightarrow & (f_0)_* \circ (i_Y^{(1)})^* \\ \downarrow & & \downarrow \\ (-)^{\mu_\infty} \circ \Psi_{\eta,p}^t \circ j_X^* \circ f_* & \longrightarrow & (-)^{\mu_\infty} \circ (\bar{f}_0)_* \circ \Psi_{\eta,q}^t \circ j_Y^* \end{array}$$

commutes.

Proof. The bottom row in the square is homotopic to the composition

$$(5.10.2) \quad i_X^* \circ (j_X)_* \circ j_X^* \circ f_* \rightarrow i_X^* \circ (j_X)_* \circ (f_U)_* \circ j_Y^* \simeq i_X^* \circ f_* \circ (j_Y)_* \circ j_Y^* \rightarrow (f_0)_* \circ i_Y^* \circ (j_Y)_* \circ j_Y^*.$$

Thus, one obtains a diagram $\partial(\Delta^1 \times \Delta^1) \rightarrow \text{Fun}(\mathcal{D}_{\text{ét}}(Y; \Lambda), \mathcal{D}_{\text{ét}}(X_0; \Lambda))$

$$(5.10.3) \quad \begin{array}{ccc} (i_X^{(1)})^* \circ f_* & \longrightarrow & (f_0)_* \circ (i_Y^{(1)})^* \\ \downarrow & & \downarrow \\ i_X^* \circ f_* \circ (j_Y)_* \circ j_Y^* & \longrightarrow & (f_0)_* \circ i_Y^* \circ (j_Y)_* \circ j_Y^*, \end{array}$$

where the vertical arrows are induced by the unit of the adjunction $(j_Y^*, (j_Y)_*)$ and the horizontal arrows by the base change morphism $(i_X^{(1)})^* \circ f_* \rightarrow (f_0)_* \circ (i_Y^{(1)})^*$. It is then clear that the two compositions are homotopic, whence the claim. \square

Therefore, the data $((i_X^{(1)})^* \circ f_* \rightarrow (f_0)_* \circ (i_Y^{(1)})^*, \Psi_{\eta,p}^t \circ j_X^* \circ f_* \rightarrow (\bar{f}_0)_* \circ \Psi_{\eta,q}^t \circ j_Y^*, \tau)$ define the morphism (5.8.1).

Proposition 5.11. *If $f : Y \rightarrow X$ is a proper morphism representable by DM stacks, then (5.8.1) is an equivalence.*

Proof. As $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ and $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0)^{\mu\infty}$ detect equivalences, it suffices to show that $(i_X^{(1)})^* \circ f_* \rightarrow (f_0)_* \circ (i_Y^{(1)})^*$ and $\Psi_{\eta,p}^t \circ j_X^* \circ f_* \rightarrow (\tilde{f}_0)_* \circ \Psi_{\eta,q}^t \circ j_Y^*$ are equivalences, which follows from proper base change. For the former this is immediate, while for the latter it suffices invoke point (3) in Lemma 5.9 and the proper base change equivalence $(f_U)_* \circ j_Y^* \simeq j_X^* \circ f_*$. \square

Vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ and !-pushforward. In this subsection we will define a natural transformation

$$(5.11.1) \quad (f_{\mathcal{Y}^t})_! \circ \Psi_q^t \rightarrow \Psi_p^t \circ f_!$$

for every separated morphism $f : Y \rightarrow X$ locally of finite type. Composing $(f_{\mathcal{Y}^t})_! \circ \Psi_q^t$ and $\Psi_p^t \circ f_!$ with the localization ∞ -functor $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ we get $(f_0^{(1)})_! \circ (i_Y^{(1)})^*$ and $(i_X^{(1)})^* \circ f_!$, that are equivalent ([19]). On the other hand, if we compose them with the localization ∞ -functor $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu\infty}$, we find $(\tilde{f}_0)_! \circ \Psi_{q,\eta}^t \circ j_Y^*$ and $\Psi_{p,\eta}^t \circ j_X^* \circ f_!$ respectively.

Lemma 5.12. (1) *There is an equivalence of ∞ -functors $(\tilde{f}_0)_! \circ \bar{i}_Y^* \simeq \bar{i}_X^* \circ \bar{f}_!$.*

(2) *There is a natural transformation $\gamma_f : \bar{f}_! \circ (\bar{j}_Y)_* \rightarrow (\bar{j}_X)_* \circ (f_U)_!$.*

Proof. (1) Passing to right adjoints, we will show that there is an equivalence of ∞ -functors $(\bar{i}_Y)_* \circ \bar{f}_!^1 \simeq \bar{f}_!^1 \circ (\bar{i}_X)_*$. The first ∞ -functor can be obtained as the limit of the diagram

$$(5.12.1) \quad \mathbb{N}_S^{op} \rightarrow \text{Fun}(\Delta^1, \text{Corr}(N(\mathbf{Art}_S))_{F,all}) \rightarrow \text{Fun}(\Delta^1, \mathbf{Pr}_{\text{stb}}^R),$$

where the first arrow is the diagram

$$(5.12.2) \quad n \mapsto \begin{array}{ccc} Y_0^{(n)} & \xrightarrow{f_0^{(n)}} & X_0^{(n)} \\ & \downarrow i_Y^{(n)} & \\ & Y^{(n)} & \end{array}$$

However, this diagram of correspondences is equivalent to the composition of the diagrams

$$(5.12.3) \quad n \mapsto \begin{array}{ccc} X_0^{(n)} & \xrightarrow{id} & X_0^{(n)} \\ & \downarrow i_X^{(n)} & \\ & X^{(n)} & \end{array} \quad n \mapsto \begin{array}{ccc} Y^{(n)} & \xrightarrow{f^{(n)}} & X^{(n)} \\ & \downarrow id & \\ & Y^{(n)} & \end{array}$$

whence the equivalence.

(2) By adjunction, we must provide a natural transformation $(\bar{j}_Y)_* \rightarrow \bar{f}_!^1 \circ (\bar{j}_X)_* \circ (f_U)_*$. Using an analogue argument of the one in (1), we see that $\bar{f}_!^1 \circ (\bar{j}_X)_* \simeq (\bar{j}_Y)_* \circ f_U^!$. We define the above-mentioned natural transformation as the morphism

$$(5.12.4) \quad (\bar{j}_Y)_* \rightarrow \bar{f}_!^1 \circ (\bar{j}_X)_* \circ (f_U)_* \simeq (\bar{j}_Y)_* \circ f_U^! \circ (f_U)_!$$

induced by the unit of the adjunction $((f_U)_!, f_U^!)$. \square

We define

$$(5.12.5) \quad (\bar{f}_0)! \circ \Psi_{\eta,q}^t \circ j_Y^* \rightarrow \Psi_{\eta,p}^t \circ j_X^* \circ f!$$

as the following composition:

$$(5.12.6) \quad (\bar{f}_0)! \circ \bar{i}_Y^* \circ (\bar{j}_Y)_* \circ j_Y^* \simeq \bar{i}_X^* \circ \bar{f}_! \circ (\bar{j}_Y)_* \circ j_Y^* \xrightarrow{\gamma_f} \bar{i}_X^* \circ (\bar{j}_Y)_* \circ (f_U)! \circ j_Y^* \simeq \bar{i}_X^* \circ (\bar{j}_Y)_* j_X^* \circ f!$$

Lemma 5.13. *The square $\tau : \partial(\Delta^1 \times \Delta^1) \rightarrow \text{Fun}(\mathcal{D}_{\text{ét}}(Y; \Lambda), \mathcal{D}_{\text{ét}}(X_0; \Lambda))$*

$$(5.13.1) \quad \begin{array}{ccc} (f_0^{(1)})! \circ (i_Y^{(1)})^* & \longrightarrow & (i_X^{(1)})^* \circ f! \\ \downarrow & & \downarrow \\ (-)^{\mu_{\infty,S}} \circ (\bar{f}_0)! \circ \Psi_{q,\eta}^t \circ j_Y^* & \rightarrow & (-)^{\mu_{\infty,S}} \circ \Psi_{p,\eta}^t \circ j_X^* \circ f! \end{array}$$

commutes.

Proof. The bottom row in the square is homotopic to the composition

$$(5.13.2) \quad (f_0)! \circ i_Y^* \circ (j_Y)_* \circ j_Y^* \simeq i_X^* \circ f! \circ (j_Y)_* \circ j_Y^* \rightarrow i_X^* \circ (j_X)_* \circ (f_U)! \circ j_Y^* \simeq i_X^* \circ (j_X)_* \circ j_X^* \circ f!$$

Thus, one obtains a diagram $\partial(\Delta^1 \times \Delta^1) \rightarrow \text{Fun}(\mathcal{D}_{\text{ét}}(Y; \Lambda), \mathcal{D}_{\text{ét}}(X_0; \Lambda))$

$$(5.13.3) \quad \begin{array}{ccc} (f_0)! \circ i_Y^* & \longrightarrow & i_X^* \circ f! \\ \downarrow & & \downarrow \\ (f_0)! \circ i_Y^* \circ (j_Y)_* \circ j_Y^* & \longrightarrow & i_X^* \circ (j_X)_* \circ j_X^* \circ f! \end{array}$$

where the vertical arrows are induced by the unit of the adjunctions $(j_Y^*, (j_Y)_*)$ and $(j_X^*, (j_X)_*)$, while the horizontal arrows by the base change morphism. It is then clear that the two compositions are homotopic, whence the claim. \square

The data $((f_0^{(1)})! \circ (i_Y^{(1)})^* \simeq (i_X^{(1)})^* \circ f!, (\bar{f}_0)! \circ \Psi_{q,\eta}^t \circ j_Y^* \rightarrow \Psi_{p,\eta}^t \circ j_X^* \circ f!, \tau)$ determine the morphism (5.11.1).

Vanishing cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ and !-pullback. In this subsection we will define a morphism

$$(5.13.4) \quad \Psi_q^t \circ f! \rightarrow f_{\mathcal{Y}^t}^! \circ \Psi_p^t.$$

By composing both $\Psi_q^t \circ f!$ and $f_{\mathcal{Y}^t}^! \circ \Psi_p^t$ with the localization $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)$ we find the ∞ -functors $(i_Y^{(1)})^* \circ f!$ and $(f_0)! \circ (i_X^{(1)})^*$ respectively. We consider the morphism $(i_Y^{(1)})^* \circ f! \rightarrow (f_0)! \circ (i_X^{(1)})^*$ adjoint to

$$(5.13.5) \quad f! \rightarrow (i_Y^{(1)})_* \circ (f_0)! \circ (i_X^{(1)})^* \simeq f! \circ (i_X^{(1)})_* \circ (i_X^{(1)})^*$$

induced by the unit of the adjunction $((i_X^{(1)})^*, (i_X^{(1)})_*)$.

On the other hand, if we compose $\Psi_q^t \circ f!$ and $f_{\mathcal{Y}^t}^! \circ \Psi_p^t$ with the localization $\mathcal{D}_{\text{ét}}(\mathcal{Y}_q^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_{\infty}}$ we find the ∞ -functors $\Psi_{q,\eta}^t \circ j_Y^* \circ f!$ and $(\bar{f}_0)! \circ \Psi_{p,\eta}^t \circ j_X^*$ respectively. Notice that there is a morphism of ∞ -functors $\bar{i}_Y^* \circ f! \rightarrow \bar{f}_0^! \circ \bar{i}_X^*$, adjoint to the natural transformation

$$(5.13.6) \quad f! \rightarrow (\bar{i}_Y)_* \circ \bar{f}_0^! \circ \bar{i}_X^* \simeq f! \circ (\bar{i}_X)_* \circ \bar{i}_X^*$$

induced by the unit of the adjunction $(\bar{i}_X^*, (\bar{i}_X)_*)$.

We define $\Psi_{q,\eta}^t \circ j_Y^* \circ f^! \rightarrow (\tilde{f}_0)^! \circ \Psi_{p,\eta}^t \circ j_X^*$ as the composition

$$(5.13.7) \quad \bar{i}_Y^* \circ (\bar{j}_Y)_* \circ j_Y^* \circ f^! \rightarrow \bar{i}_Y^* \circ (\bar{j}_Y)_* \circ f_U^! \circ j_X^* \simeq \bar{i}_Y^* \circ \bar{f}^! \circ (\bar{j}_X)_* \circ j_X^* \rightarrow \bar{f}_0^! \circ \bar{i}_X^* \circ (\bar{j}_X)_* \circ j_X^*.$$

Lemma 5.14. *The square $\tau : \partial(\Delta^1 \times \Delta^1) \rightarrow \text{Fun}(\mathcal{D}_{\text{ét}}(Y; \Lambda), \mathcal{D}_{\text{ét}}(X_0; \Lambda))$*

$$(5.14.1) \quad \begin{array}{ccc} (i_Y^{(1)})^* \circ f^! & \longrightarrow & (f_0)^! \circ (i_X^{(1)})^* \\ \downarrow & & \downarrow \\ (-)^{\mu_\infty, S} \circ \Psi_{q,\eta}^t \circ j_Y^* \circ f^! & \rightarrow & (-)^{\mu_\infty, S} \circ (\tilde{f}_0)^! \circ \Psi_{p,\eta}^t \circ j_X^* \end{array}$$

commutes.

Proof. This follows from the functoriality of the exchange morphisms $Ex^{*,!}$. \square

We obtain 5.13.4 by applying Corollary 2.7 to the datum $((i_Y^{(1)})^* \circ f^! \rightarrow (f_0)^! \circ (i_X^{(1)})^*, \Psi_{q,\eta}^t \circ j_Y^* \circ f^! \rightarrow (\tilde{f}_0)^! \circ \Psi_{p,\eta}^t \circ j_X^*, \tau)$.

6. COMPARISON WITH VANISHING CYCLES OVER A STRICTLY HENSELIAN TRAIT

Assume that S is a strictly henselian trait and keep the notation of section §3. Let $p : X \rightarrow S$ a morphism of finite type. The uniformizer π induces a morphism $\pi : S \rightarrow \mathbb{A}_S^1$. Let $f : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ denote the following composition:

$$(6.0.1) \quad X \xrightarrow{p} S \xrightarrow{\pi} \mathbb{A}_S^1 \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}].$$

It is natural to compare the theory of tame vanishing cycles of f with that of p (defined in [11, Exposé XIII]). First observe that these two ∞ -functors are indeed comparable. Notice that $X_0 \simeq X_\sigma$.

Lemma 6.1. *There exists a commutative square in $\mathbf{Pr}_{\text{stb}}^{\mathbf{R}}$*

$$(6.1.1) \quad \begin{array}{ccc} \mathcal{D}_{\text{ét}}(X_0; \Lambda) & \xleftarrow{(-)^{\mu_\infty}} & \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \\ \downarrow id & & \downarrow \simeq \\ \mathcal{D}_{\text{ét}}(X_0; \Lambda) & \xleftarrow{(-)^{I^t}} & \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{I^t}. \end{array}$$

In particular, there is an equivalence $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \simeq \mathcal{D}_{\text{ét}}(\mathcal{Y}_f^t; \Lambda)$.

Proof. First notice that, since we assume that S is strictly henselian, $\text{Gal}(k^s/k) \simeq 0$. In particular, $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda)$ is the recollement of $\mathcal{D}_{\text{ét}}(X_0; \Lambda)$ and $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ and the second statement follows immediately from the first.

We need to show that for every $n \in \mathbb{N}_S$ there is a commutative square

$$(6.1.2) \quad \begin{array}{ccc} \mathcal{D}_{\text{ét}}(X_0; \Lambda) & \xleftarrow{(-)^{\mu_n}} & \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n} \\ \downarrow \text{id} & & \downarrow \simeq \\ \mathcal{D}_{\text{ét}}(X_0; \Lambda) & \xleftarrow{(-)^{\text{Gal}(K[\pi^{\frac{1}{n}}]/K)}} & \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\text{Gal}(K[\pi^{\frac{1}{n}}]/K)}. \end{array}$$

Moreover, these squares have to fit in a diagram $\mathbb{N}_S \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathbf{Pr}_{\text{stb}}^{\text{R}})$.

Since S is strictly henselian, the group scheme μ_n is isomorphic to $\text{Gal}(K[\pi^{\frac{1}{n}}]/K) \times S$, where the scheme on the right denotes the group scheme $\prod_{\text{Gal}(K[\pi^{\frac{1}{n}}]/K)} S$ with the group structure induced by $\text{Gal}(K[\pi^{\frac{1}{n}}]/K)$. In particular, there are equivalences of DM stacks

$$(6.1.3) \quad X_0^{(n)} \simeq [X/\text{Gal}(K[\pi^{\frac{1}{n}}]/K)] \times_X X_0.$$

Moreover, they are compatible in the obvious sense, i.e. there is an equivalence of diagrams of DM stacks

$$(6.1.4) \quad X_0^{(\bullet)} \simeq [X/\text{Gal}(K[\pi^{\frac{1}{\bullet}}]/K)] \times_X X_0 : \mathbb{N}_S \rightarrow \mathbf{DM}_S.$$

The claim follows immediately. \square

Theorem 6.2. *There is an equivalence of ∞ -functors*

$$(6.2.1) \quad \Psi_p^{cl,t} \simeq \Psi_f^t : \mathcal{D}_{\text{ét}}(X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \simeq \mathcal{D}_{\text{ét}}(\mathcal{Y}_f^t; \Lambda).$$

Proof. As the compositions of $\Psi_p^{cl,t}$ and Ψ_f^t with $\mathcal{D}_{\text{ét}}(\mathcal{Y}_p^t; \Lambda) \simeq \mathcal{D}_{\text{ét}}(\mathcal{Y}_f^t; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ clearly agree, it suffices to show that there is an equivalence $\Psi_{\eta,p}^{cl,t} \circ j_X^* \simeq \Psi_{\eta,f}^t \circ j_X^*$, compatible with the morphisms $i_X^* \rightarrow \Psi_{\eta,p}^{cl,t} \circ j_X^*$ and $i_X^* \rightarrow \Psi_{\eta,f}^t \circ j_X^*$. This relies on the fact that, as we said above, S is strictly henselian and for each $n \in \mathbb{N}_S$ there is an isomorphism of group S -schemes

$$(6.2.2) \quad \mu_n \simeq \text{Gal}(K[\pi^{\frac{1}{n}}]/K) \times S.$$

Therefore we get equivalences of diagrams of DM stacks $\mathbb{N}_S \times (\cdot \rightarrow \cdot \leftarrow \cdot) \rightarrow \mathbf{DM}_S$

$$(6.2.3) \quad \begin{array}{ccccc} X_0^{(\bullet)} & \xrightarrow{i_X^{(\bullet)}} & X^{(\bullet)} & \xleftarrow{j_X^{(\bullet)}} & X_\eta \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ [X/\text{Gal}(K[\pi^{\frac{1}{\bullet}}]/K)] \times_X X_0 & \xrightarrow{i_X^{(\bullet)}} & [X/\text{Gal}(K[\pi^{\frac{1}{\bullet}}]/K)] & \xleftarrow{j_X^{(\bullet)}} & X_\eta \end{array}$$

Then the theorem follows from the comparison between the classical definition of tame nearby cycles and that using DM stacks carried out in Section §3 (Proposition 3.15). \square

7. TAME NEARBY CYCLES OVER \mathbb{A}_S^1 AND COMPARISON WITH THE ÉTALE VERSION OF AYOUB'S TAME NEARBY CYCLES

In [3, 4] J. Ayoub introduced a formalism of tame nearby cycles over \mathbb{A}_S^1 in the motivic context, which he proved to be compatible with tame nearby cycles under étale/ ℓ -adic realization in [5]. Obviously, its construction also makes sense in the étale setting. Another formalism of tame nearby cycles over

\mathbb{A}_S^1 is provided by the formalism introduced here: for every $f : X \rightarrow \mathbb{A}_S^1$, we can apply our formalism to the composition $p : X \xrightarrow{f} \mathbb{A}_S^1 \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$. In this section we will prove that these two notions of tame nearby cycles over \mathbb{A}_S^1 agree in a suitable sense. In order to do so, we shall recall Ayoub's construction, following [5]. Let $(\mathcal{E}_S, \mathbb{N}_S)$ denote the following diagram of S -schemes:

$$(7.0.1) \quad \begin{array}{c} \mathcal{E}_S : \mathbb{N}_S \rightarrow \mathbf{Sch}_S \\ (n \rightarrow nm) \mapsto \mathbb{G}_{m,S} \xrightarrow{(-)^m} \mathbb{G}_{m,S}. \end{array}$$

Notation 7.1. For an S -scheme Z , let (Z, \mathbb{N}_S) denote the constant diagram indexed by \mathbb{N}_S with constant value Z .

Remark 7.2. Notice that there is a natural transformation $(\mathcal{E}_S, \mathbb{N}_S) \rightarrow (\mathbb{G}_{m,S}, \mathbb{N}_S)$ which corresponds to $(n \in \mathbb{N}_S)$

$$(7.2.1) \quad \mathcal{E}_S(n) = \mathbb{G}_{m,S} \xrightarrow{(-)^n} \mathbb{G}_{m,S}.$$

For $f : X \rightarrow S$, in *loc. cit.* Ayoub considers the following diagram (of \mathbb{N}_S -diagrams):

$$(7.2.2) \quad \begin{array}{ccccccc} (X_0, \mathbb{N}_S) & \xrightarrow{i_X} & (X, \mathbb{N}_S) & \xleftarrow{j_X} & (U_X, \mathbb{N}_S) & \xleftarrow{v_X} & (\mathcal{E}_f, \mathbb{N}_S) \\ \downarrow f_0 & & \downarrow f & & \downarrow f_U & & \downarrow f_{\mathcal{E}} \\ (S, \mathbb{N}_S) & \xrightarrow{i} & (\mathbb{A}_S^1, \mathbb{N}_S) & \xleftarrow{j} & (\mathbb{G}_{m,S}, \mathbb{N}_S) & \xleftarrow{v} & (\mathcal{E}_S, \mathbb{N}_S). \end{array}$$

All squares are Cartesian. Then Ayoub defines the following ∞ -functor (see [5, Formula (100), pag. 69]):

$$(7.2.3) \quad \Psi_{f,\eta}^{Ay} := (p_0)_\# \circ i_X^* \circ j_{X,*} \circ v_{X,*} \circ v_X^* \circ p_U^* : \mathcal{D}_{\text{ét}}(U_X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

Here, $p_0 : (X_0, \mathbb{N}_S) \rightarrow (X_0, \Delta^0)$ (resp. $p_U : (U_X, \mathbb{N}_S) \rightarrow (U_X, \Delta^0)$) denotes the obvious morphism of diagrams of schemes. Notice that the diagram $(\mathcal{E}_f, \mathbb{N}_S)$ is $n \mapsto U_X \times_{\mathbb{G}_{m,S}} \mathbb{G}_{m,S}$, where we consider the pullback along the morphism $\mathbb{G}_{m,S} \xrightarrow{(-)^n} \mathbb{G}_{m,S}$.

Let $\text{Forget} : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ be the left adjoint to the ∞ -functor $\mathcal{D}_{\text{ét}}(X_0; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty}$ defined by the pushforwards along the canonical morphisms $X_0 \rightarrow X_0^{(n)}$. Then we have the following

Theorem 7.3. *There is an equivalence of ∞ -functors*

$$(7.3.1) \quad \Psi_{f,\eta}^{Ay} \simeq \text{Forget} \circ \Psi_{p,\eta}^t : \mathcal{D}_{\text{ét}}(U_X; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

Proof. As (X_0, \mathbb{N}_S) , (X, \mathbb{N}_S) and (U_X, \mathbb{N}_S) are constant diagrams of S -schemes, there are equivalences of ∞ -categories

$$(7.3.2) \quad \mathcal{D}_{\text{ét}}((X_0, \mathbb{N}_S); \Lambda) \simeq \mathcal{D}_{\text{ét}}(X_0; \Lambda), \quad \mathcal{D}_{\text{ét}}((X, \mathbb{N}_S); \Lambda) \simeq \mathcal{D}_{\text{ét}}(X; \Lambda), \quad \mathcal{D}_{\text{ét}}((U_X, \mathbb{N}_S); \Lambda) \simeq \mathcal{D}_{\text{ét}}(U_X; \Lambda).$$

This means that, for any $\mathcal{F} \in \mathcal{D}_{\text{ét}}(U_X; \Lambda)$, by definition we have that

$$(7.3.3) \quad \Psi_{f,\eta}^{Ay}(\mathcal{F}) \simeq \varinjlim_{n \in \mathbb{N}_S} i_X^* \circ j_{X,*}^{(n)} \circ (v_X^{(n)})^*(\mathcal{F}),$$

where $i_X : X_0 \rightarrow X$, $j_X^{(n)} : U_X \times_{\mathbb{G}_{m,S}} \mathbb{G}_{m,S} = \mathcal{E}_f(n) \rightarrow X$ and $v_X^{(n)} : \mathbb{G}_{m,S} = \mathcal{E}_f(n) \rightarrow \mathbb{G}_{m,S}$. Notice that these morphisms naturally fit in the following diagram

$$(7.3.4) \quad \begin{array}{ccccccc} X_0 & \xrightarrow{i_X} & X & \xleftarrow{j_X} & U_X & \xleftarrow{v_X^{(n)}} & \mathcal{E}_f(n) \\ \downarrow q_0 & & \downarrow q & & \downarrow q_U & & \downarrow q_U^{(n)} \\ X_0^{(n)} & \xrightarrow{i_X^{(n)}} & X^{(n)} & \xleftarrow{j_X} & U_X^{(n)} = X^{(n)} \times_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]} S & \xleftarrow{v_X^{(n)}} & U_X^{(n)} \end{array}$$

(Note: In the original image, there are additional arrows: a top arrow $j_X^{(n)} : X \rightarrow U_X$, a right arrow $v_X^{(n)} : U_X \rightarrow \mathcal{E}_f(n)$, and a bottom arrow $j_X^{(n)} : X^{(n)} \rightarrow U_X^{(n)}$. The diagram is a commutative square with a top row and a bottom row, and vertical maps between them. The top row is $X_0 \xrightarrow{i_X} X \xleftarrow{j_X} U_X \xleftarrow{v_X^{(n)}} \mathcal{E}_f(n)$. The bottom row is $X_0^{(n)} \xrightarrow{i_X^{(n)}} X^{(n)} \xleftarrow{j_X} U_X^{(n)} = X^{(n)} \times_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]} S \xleftarrow{v_X^{(n)}} U_X^{(n)}$. Vertical maps are $q_0 : X_0 \rightarrow X_0^{(n)}$, $q : X \rightarrow X^{(n)}$, $q_U : U_X \rightarrow U_X^{(n)}$, and $q_U^{(n)} : \mathcal{E}_f(n) \rightarrow U_X^{(n)}$. There are also top arrows $j_X^{(n)} : X \rightarrow U_X$ and $j_X^{(n)} : X^{(n)} \rightarrow U_X^{(n)}$. The diagram is labeled (7.3.4).

where all squares are Cartesian and all vertical morphisms are smooth. Since Forget is a left adjoint,

$$(7.3.5) \quad \text{Forget} \circ \Psi_{p,\eta}^t(\mathcal{F}) \simeq \varinjlim_{n \in \mathbb{N}_S} \text{Forget}((i_X^{(n)})^* \circ (j_X^{(n)})_* \circ (v_X^{(n)})^*(\mathcal{F})).$$

By definition, $\text{Forget} : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_n} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ corresponds to q_0^* . Moreover, $X_0 \rightarrow X_0^{(n)} \times_{X^{(n)}} X$ is an homeomorphism. Then, by the smooth base change theorem, we get that

$$(7.3.6) \quad q_0^* \circ (i_X^{(n)})^* \circ (j_X^{(n)})_* \circ (v_X^{(n)})^*(\mathcal{F}) \simeq i_X^* \circ j_{X,*}^{(n)} \circ (v_X^{(n)})^*(\mathcal{F}).$$

In particular, this implies that

$$(7.3.7) \quad \text{Forget} \circ \Psi_{p,\eta}^t(\mathcal{F}) \simeq \Psi_{f,\eta}^{Ay}(\mathcal{F}).$$

Naturality in \mathcal{F} follows immediately from the fact that all the passages we have used are canonical. \square

8. COMPATIBILITY WITH TENSOR PRODUCT AND DUALITY

Notice that since the forgetful functor $\text{CAlg}(\mathbf{Pr}_{\text{stb}}^{\text{R}}) \rightarrow \mathbf{Pr}_{\text{stb}}^{\text{R}}$ is a right adjoint, and for any derived stack Z the ∞ -category $\mathcal{D}_{\text{ét}}(Z; \Lambda)$ is symmetric monoidal, for every $Z \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, the ∞ -categories $\mathcal{D}_{\text{ét}}(Z; \Lambda)^{\mu_\infty}$ and $\mathcal{D}_{\text{ét}}(Z_0; \Lambda)^{\mu_\infty}$ are symmetric monoidal (see [21, Corollary 3.2.2.5] for the existence of limits in $\text{CAlg}(\mathbf{Pr}_{\text{stb}}^{\text{R}})$ and for the fact that they are preserved under the forgetful functor). More precisely, these ∞ -categories are *closed* symmetric monoidal. In fact, we can equivalently define them in the ∞ -category $\text{CAlg}(\mathbf{Pr}_{\text{stb}}^{\text{L}})$. Let $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ be an Artin stack over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$. In this section we will prove some compatibilities of $\Psi_{p,\eta}^t$ with external tensor products and with duality, using the compatibility of tame nearby cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ with the étale version of Ayoub's tame nearby cycles, for which such compatibilities are already established (see [3, 4, 5, 15]).

Künneth. In this subsection we shall define the Künneth morphism in our context and prove its compatibility with tame nearby cycles in characteristic zero.

Notation 8.1. In this section we will consider two morphisms $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, $q : Y \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ of Artin stacks. Let $r : Z := X \times_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]} Y \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$.

Definition 8.2. We define the μ_∞ -equivariant external tensor product $-\boxtimes^{\mu_\infty}-$ as the composition

$$(8.2.1) \quad \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu_\infty} \times \mathcal{D}_{\text{ét}}(Y_0; \Lambda)^{\mu_\infty} \xrightarrow{(\bar{p}^r_{X_0})^* \times (\bar{p}^r_{Y_0})^*} \mathcal{D}_{\text{ét}}(Z_0; \Lambda)^{\mu_\infty} \times \mathcal{D}_{\text{ét}}(Z_0; \Lambda)^{\mu_\infty} \xrightarrow{-\boxtimes-} \mathcal{D}_{\text{ét}}(Z_0; \Lambda)^{\mu_\infty},$$

where $(\bar{p}r_{X_0})^*$ (resp. $(\bar{p}r_{Y_0})^*$) is the ∞ -functor of Construction 5.2 and $- \otimes -$ is the tensor product on $\mathcal{D}_{\text{ét}}(Z_0; \Lambda)^{\mu_\infty}$.

We are now ready to define the *Künneth morphism*. Let $F \in \mathcal{D}_{\text{ét}}(U_X; \Lambda)$ and $G \in \mathcal{D}_{\text{ét}}(U_Y; \Lambda)$. We have canonical morphisms

$$(8.2.2) \quad (\bar{p}r_{X_0})^* \Psi_{\eta,p}^t(F) \rightarrow \Psi_{\eta,r}^t((pr_{U_X})^* F), \quad (\bar{p}r_{Y_0})^* \Psi_{\eta,q}^t(G) \rightarrow \Psi_{\eta,r}^t((pr_{U_Y})^* G)$$

as constructed in Section 5. Therefore, we can consider the morphism in $\mathcal{D}_{\text{ét}}(Z_0; \Lambda)^{\mu_\infty}$

$$(8.2.3) \quad \Psi_{\eta,p}^t(F) \boxtimes^{\mu_\infty} \Psi_{\eta,q}^t(G) \rightarrow \Psi_{\eta,r}^t((pr_{U_X})^* F) \otimes \Psi_{\eta,r}^t((pr_{U_Y})^* G).$$

Being a composition of lax monoidal ∞ -functors, $\Psi_{\eta,r}^t$ is lax monoidal. Therefore, there is a canonical morphism

$$(8.2.4) \quad \Psi_{\eta,r}^t((pr_{U_X})^* F) \otimes \Psi_{\eta,r}^t((pr_{U_Y})^* G) \rightarrow \Psi_{\eta,r}^t(F \boxtimes G),$$

where $F \boxtimes G$ denotes the external tensor product

$$(8.2.5) \quad \mathcal{D}_{\text{ét}}(U_X; \Lambda) \times \mathcal{D}_{\text{ét}}(U_Y; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(U_Z; \Lambda).$$

Definition 8.3. Let $F \in \mathcal{D}_{\text{ét}}(U_X; \Lambda)$ and $G \in \mathcal{D}_{\text{ét}}(U_Y; \Lambda)$. We define the *Künneth morphism* (for F and G) as the composition of the two morphisms above:

$$(8.3.1) \quad \text{Kü} : \Psi_{\eta,p}^t(F) \boxtimes^{\mu_\infty} \Psi_{\eta,q}^t(G) \rightarrow \Psi_{\eta,r}^t(F \boxtimes G).$$

Theorem 8.4. *Assume that S is the spectrum of an algebraically closed field of characteristic 0. Let $F \in \mathcal{D}_{\text{ét}}(U_X; \Lambda)$ and let $G \in \mathcal{D}_{\text{ét}}(U_Y; \Lambda)$ be constructible objects. The Künneth morphism*

$$(8.4.1) \quad \text{Kü} : \Psi_{p,\eta}^t(F) \boxtimes \Psi_{q,\eta}^t(G) \rightarrow \Psi_{p,\eta}^t(F \boxtimes G)$$

is an equivalence.

Proof. Choose any smooth surjective morphism $X' \rightarrow X$, with X' a scheme. The compatibility of external tensor product with pullbacks and that of tame nearby vanishing cycles with pullbacks along smooth morphisms tell us that it suffices to consider the Künneth morphism for X' . Therefore, we might assume that X is a scheme.

Since the property of being an equivalence is local on X_0 with respect to the Zariski topology and the Künneth morphism is compatible with pullbacks along smooth morphisms (in particular, along open embeddings), we might assume that $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ factors through $f : X \rightarrow \mathbb{A}_S^1$. Also, forgetting the continuous action of μ_∞ is conservative and compatible with external products and therefore we might ignore it. In this case, we are considering Ayoub's tame nearby cycles over \mathbb{A}_S^1 (étale version) by Theorem 7.3. The theorem follows by [4, Théorème 3.5.17] and by the results of [5]. \square

Duality. In this subsection we will investigate how tame nearby cycles over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ behave with respect to duality. Let $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ be a Artin stack over $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$. Let $p_U : X \times_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]} S = U_X \rightarrow S$.

Assume that p is separated of finite type. Recall that the *duality functor* on $\mathcal{D}_{\text{ét}}(U_X; \Lambda)$ is

$$(8.4.2) \quad \mathbb{D}_{U_X} := \underline{\text{Hom}}_{U_X}(-, p_U^!(\underline{\Lambda}_S)) : \mathcal{D}_{\text{ét}}(U_X; \Lambda)^{\text{op}} \rightarrow \mathcal{D}_{\text{ét}}(U_X; \Lambda),$$

where $\underline{\text{Hom}}_{U_X}$ denotes the internal hom in $\mathcal{D}_{\text{ét}}(U_X; \Lambda)$.

Similarly, we define the *duality functor on $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu\infty}$* as the ∞ -functor

$$(8.4.3) \quad \mathbb{D}_{X_0} := \underline{\text{Hom}}(-, \bar{p}_0^!(\underline{\Lambda}_{\text{BG}_{m,S}})) : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu\infty, \text{op}} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu\infty}.$$

Let $A \in \mathcal{D}_{\text{ét}}(U_X; \Lambda)$. There is a canonical morphism

$$(8.4.4) \quad \Psi_{p,\eta}^t(\mathbb{D}_{U_X}(A)) \rightarrow \mathbb{D}_{X_0}(\Psi_{p,\eta}^t(A)),$$

defined by adjunction by the composition

$$(8.4.5) \quad \Psi_{p,\eta}^t(\mathbb{D}_{U_X}(A)) \otimes \Psi_{p,\eta}^t(A) \rightarrow \Psi_{p,\eta}^t(\mathbb{D}_{U_X}(A) \otimes A) \rightarrow \Psi_{p,\eta}^t(p_U^!(\underline{\Lambda}_S)) \rightarrow \bar{p}_0^!(\Psi_{id,\eta}^t(\underline{\Lambda}_S)) \simeq \bar{p}_0^!(\underline{\Lambda}_{\text{BG}_{m,S}}),$$

where the first arrow is induced by the lax monoidal structure on $\Psi_{p,\eta}^t$, the second arrow by the canonical map $\mathbb{D}_{U_X}(A) \otimes A \rightarrow p_U^!(\underline{\Lambda}_S)$ and the third arrow by the map constructed in Section 5. The equivalence

$$(8.4.6) \quad \Psi_{id,\eta}^t(\underline{\Lambda}_S) \simeq \underline{\Lambda}_{\text{BG}_{m,S}}$$

will be proved in Section 9.

Theorem 8.5. *Let S be the spectrum of a field. Assume that A is a constructible object in $\mathcal{D}_{\text{ét}}(U_X; \Lambda)$. Then*

$$(8.5.1) \quad \Psi_{p,\eta}^t(\mathbb{D}_{U_X}(A)) \simeq \mathbb{D}_{X_0}(\Psi_{p,\eta}^t(A)).$$

Proof. The question is local on X , so that we consider the pullback along an open embedding $j : V \rightarrow X$. Indeed, the left hand side commutes with pullbacks along smooth morphisms. For what concerns the right hand side, it is easy to see (using the adjunctions $(\bar{j}_0!, \bar{j}_0^!)$, $(\bar{j}_0^*, \bar{j}_0*)$ and the equivalence $\bar{j}_0^* \simeq \bar{j}_0^!$) that

$$(8.5.2) \quad \bar{j}_0^* \circ \mathbb{D}_{X_0}(\Psi_{p,\eta}^t(A)) \simeq \mathbb{D}_{V_0}(\Psi_{p \circ j, \eta}^t(A|_V)).$$

Therefore, we can assume without loss of generality that $p : X \rightarrow [\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ factors through \mathbb{A}_S^1 .

Moreover, as *Forget* : $\mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\mu\infty} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ is a conservative functor, it suffices to show that the induced map

$$(8.5.3) \quad \text{Forget} \circ \Psi_{p,\eta}^t(\mathbb{D}_{U_X}(A)) \rightarrow \text{Forget} \circ \mathbb{D}_{X_0}(\Psi_{p,\eta}^t(A))$$

is an equivalence. By Theorem 7.3, the left hand side is the (étale version) of Ayoub's nearby cycles. As for the right hand side, we claim that

$$(8.5.4) \quad \text{Forget} \circ \mathbb{D}_{X_0} \simeq \mathbb{D}_{X_0} \circ \text{Forget},$$

where \mathbb{D}_{X_0} stands for $\underline{\text{Hom}}_{X_0}(-, p_0^!(\underline{\Lambda}_{\text{BG}_{m,S}})) : \mathcal{D}_{\text{ét}}(X_0; \Lambda)^{\text{op}} \rightarrow \mathcal{D}_{\text{ét}}(X_0; \Lambda)$ on the right hand side of the formula. However, this follows immediately from the fact that *Forget* is induced by the pullback along the smooth atlases $X_0 \rightarrow X_0^{(n)}$ (which are smooth morphisms of relative dimension 0) and from the fact that duality exchanges $*$ -pullbacks with $!$ -pullbacks and viceversa. Therefore, we are reduced to show that

$$\text{Forget} \circ \Psi_{p,\eta}^t(\mathbb{D}_{U_X}(A)) \rightarrow \mathbb{D}_{X_0}(\text{Forget} \circ \Psi_{p,\eta}^t(A))$$

is an equivalence, which follows immediately from Theorem 7.3 and [3, 4, Théorème 3.5.20], [5] and [15, Theorem 3.2.5] \square

9. TAME VANISHING CYCLES OVER $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$ AND MONODROMY INVARIANT VANISHING CYCLES

In [25], we introduced the so called *monodromy invariant vanishing cycles* (in the ℓ -adic setting). Let us briefly recall the definition. Let X be an S scheme. Fix a line bundle \mathcal{L} on X and a global section $s \in H^0(X, \mathcal{L})$. Consider the diagram

$$(9.0.1) \quad \begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{j} & U_X \\ \downarrow s_0 & & \downarrow s & & \downarrow s_U \\ X & \xrightarrow{i_0} & \mathbb{V}(\mathcal{L}) & \xleftarrow{j_0} & \mathcal{U} = \mathbb{V}(\mathcal{L}) - X \end{array}$$

where $\mathbb{V}(\mathcal{L}) = \text{Spec}_X(\text{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee))$ is the total space of \mathcal{L} , $s : X \rightarrow \mathbb{V}(\mathcal{L})$ is the global section determined by s and both squares are Cartesian.

Definition 9.1. For an étale sheaf $\mathcal{F} \in \mathcal{D}_{\text{ét}}(\mathcal{U}; \Lambda)$, we define *monodromy invariant vanishing cycles with coefficients in \mathcal{F}* as

$$(9.1.1) \quad \Phi_{(X,s)}^{\text{mi}}(\mathcal{F}) := \text{cofiber}(i^* s^* j_{0*} \mathcal{F} \rightarrow i^* j_* s_U^* \mathcal{F}) \in \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

In [25], it is proved that ³

$$(9.1.2) \quad \Phi_{(X,s)}^{\text{mi}}(\underline{\Lambda}_X) \simeq \text{cofiber}\left(\text{cofiber}(c_1(\mathcal{L}|_{X_0}) : \underline{\Lambda}_{X_0}(-1)[-2] \rightarrow \underline{\Lambda}_{X_0}) \rightarrow i^* j_* \underline{\Lambda}_{U_X}\right),$$

where the map $\text{cofiber}(c_1(\mathcal{L}|_{X_0}) : \underline{\Lambda}_{X_0}(-1)[-2] \rightarrow \underline{\Lambda}_{X_0}) \rightarrow i^* j_* \underline{\Lambda}_{U_X}$ is induced by the commutative diagram

$$(9.1.3) \quad \begin{array}{ccc} \underline{\Lambda}_{X_0}(-1)[-2] & \xrightarrow{c_1(\mathcal{L}|_{X_0})} & \underline{\Lambda}_{X_0} \\ \downarrow & & \downarrow \\ i^* j_* \underline{\Lambda}_{U_X}(-1)[-2] & \xrightarrow{i^* j_*(c_1(\mathcal{L}|_{U_X})) \sim 0} & i^* j_* \underline{\Lambda}_{U_X} \end{array}$$

Notice that the squares in the diagram

$$(9.1.4) \quad \begin{array}{ccccc} X_0 & \xrightarrow{i} & X & \xleftarrow{j} & U_X \\ \downarrow p_0 & & \downarrow p & & \downarrow p_U \\ \text{BG}_{m,S} & \xrightarrow{i_0} & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \xleftarrow{j_0} & S \end{array}$$

are Cartesian as well. Here p is the morphism associated to (\mathcal{L}, s) and p_0 the morphism associated to $\mathcal{L}|_{X_0}$.

We have already observed (Remark 4.16) that

$$(9.1.5) \quad (\Psi_{\eta,p}^t(\underline{\Lambda}_X))^{\mu_\infty} \simeq i^* j_* \underline{\Lambda}_{U_X}.$$

³actually, we proved what follows in the ℓ -adic setting, but the proofs work *mutatis mutandis* with finite coefficients too.

We will now show that

$$(9.1.6) \quad (\text{triv}(\underline{\Lambda}_{X_0}))^{\mu_\infty} \simeq \text{cofiber}(c_1(\mathcal{L}|_{X_0}) : \underline{\Lambda}_{X_0}(-1)[-2] \rightarrow \underline{\Lambda}_{X_0}).$$

The base. In this subsection we will investigate the construction of tame vanishing cycles applied to the identity morphism on $[\mathbb{A}_S^1/\mathbb{G}_{m,S}]$, i.e. the ∞ -functor

$$(9.1.7) \quad \Psi_{id}^t : \mathcal{D}_{\text{ét}}([\mathbb{A}_S^1/\mathbb{G}_{m,S}]; \Lambda) \rightarrow \mathcal{D}_{\text{ét}}(\mathcal{Y}_{id}^t, \Lambda).$$

For any $n, m \in \mathbb{N}_S$ there are adjunctions

$$(9.1.8) \quad \text{triv}_n^\infty : \mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_n} \rightleftarrows \mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda) : (-)^{n \cdot \mu_\infty},$$

$$(9.1.9) \quad \text{triv}_n^{nm} : \mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_n} \rightleftarrows \mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_{nm}} : (-)^{m \cdot \mu_{nm}}.$$

These ∞ -functors are compatible in the following sense:

$$(9.1.10) \quad \text{triv}_{nm}^\infty \circ \text{triv}_n^{nm} \simeq \text{triv}_n^\infty,$$

$$(9.1.11) \quad (-)^{m \cdot \mu_{nm}} \circ (-)^{nm \cdot \mu_\infty} \simeq (-)^{n \cdot \mu_\infty}.$$

By definition, $\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})$ is the object of $\mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_\infty} = \varprojlim_n \mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_n}$ corresponding to $\{(\Psi_{id,\eta}^t)^{(n)}(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})\}$. Using the above mentioned equivalences of ∞ -functors, for any $n, m \in \mathbb{N}_S$ there is a morphism

$$(9.1.12) \quad \text{triv}_n^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{n \cdot \mu_\infty}) \rightarrow \text{triv}_{nm}^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{nm \cdot \mu_\infty}))$$

induced by the isomorphism

$$(9.1.13) \quad \text{triv}_n^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{n \cdot \mu_\infty}) \simeq \text{triv}_{nm}^\infty(\text{triv}_n^{nm}((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{nm \cdot \mu_\infty})^{\mu_m}))$$

and the morphism

$$(9.1.14) \quad \text{triv}_{nm}^\infty(\text{triv}_n^{nm}((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{nm \cdot \mu_\infty})^{\mu_m})) \rightarrow \text{triv}_{nm}^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{nm \cdot \mu_\infty}))$$

induced by the counit of the adjunction $(\text{triv}_n^{nm}, (-)^{\mu_m})$. As these counits and the equivalences

$$(9.1.15) \quad \text{triv}_{nm}^{nmp} \circ \text{triv}_m^{nm} \simeq \text{triv}_m^{nmp}, \quad (-)^{\mu_m} \circ (-)^{\mu_n} \simeq (-)^{\mu_{nm}}$$

are natural, these morphisms (9.1.12) assemble in a diagram

$$(9.1.16) \quad \mathbb{N}_S \rightarrow \mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_\infty}.$$

Lemma 9.2. *The following equivalence holds in $\mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_\infty}$:*

$$(9.2.1) \quad \text{triv}(\underline{\Lambda}_{\text{BG}_{m,S}}) \simeq \varinjlim \left(\text{triv}_n^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_S))^{n \cdot \mu_\infty}) \rightarrow \text{triv}_{nm}^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_S))^{nm \cdot \mu_\infty}) \right).$$

Proof. By definition, we have that $(\Psi_{id,\eta}^t(\underline{\Lambda}_S))^{n \cdot \mu_\infty} = (\Psi_{id,\eta}^t)^{(n)}(\underline{\Lambda}_S)$. Consider the following diagram of Artin stacks

$$(9.2.2) \quad \begin{array}{ccccc} & & & i & \\ & \xrightarrow{\beta} & & \searrow & \\ \mathrm{BG}_{m,S} & \xrightarrow{\beta} & \mathrm{BG}_{m,S} \times_{i, [\mathbb{A}_S^1/\mathbb{G}_{m,S}], \Theta^{(n)}} [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \longrightarrow & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \longleftarrow & S \\ & \searrow \Theta_0^{(n)} & \downarrow & & \downarrow \Theta^{(n)} & & \downarrow id_S \\ & & \mathrm{BG}_{m,S} & \xrightarrow{i} & [\mathbb{A}_S^1/\mathbb{G}_{m,S}] & \longleftarrow & S \\ & & & & & & j \end{array}$$

Both squares are Cartesian and β is a closed embedding with empty open complementary.

$$(9.2.3) \quad (\Psi_{id,\eta}^t)^{(n)}(\underline{\Lambda}_S) := i^* j_*(\underline{\Lambda}_S)$$

Then the distinguished triangle

$$(9.2.4) \quad i^!(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]}) \rightarrow \underline{\Lambda}_{\mathrm{BG}_{m,S}} \rightarrow i^* j_*(\underline{\Lambda}_S)$$

and absolute purity for Artin stacks ([16, Theorem 3.28])

$$(9.2.5) \quad \underline{\Lambda}_{\mathrm{BG}_{m,S}}(-1)[-2] \simeq i^!(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})$$

imply that

$$(9.2.6) \quad i^* j_*(\underline{\Lambda}_S) \simeq \mathit{cofiber}(\underline{\Lambda}_{\mathrm{BG}_{m,S}}(-1)[-2] \xrightarrow{c_1} \underline{\Lambda}_{\mathrm{BG}_{m,S}}).$$

Here $c_1 \in H_{\text{ét}}^2(\mathrm{BG}_{m,S}, \underline{\Lambda}_{\mathrm{BG}_{m,S}}(1))$ is the *universal first Chern class*. Since $\Theta_0^{(n)} : \mathrm{BG}_{m,S} \rightarrow \mathrm{BG}_{m,S}$ corresponds to $n \cdot U$, where U is the *universal line bundle*, we see that

$$(9.2.7) \quad (\Theta_0^{(n)})^* \circ (\Theta_0^{(n)})_*(c_1) \simeq n \cdot c_1 : \underline{\Lambda}_{\mathrm{BG}_{m,S}}(-1)[-2] \rightarrow \underline{\Lambda}_{\mathrm{BG}_{m,S}}.$$

Under these equivalences, the morphisms $\mathit{triv}_n^{nm}((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{n \cdot \mu_\infty}) \rightarrow ((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{nm \cdot \mu_\infty})$ corresponds to ([5, Proposition 10.3])

$$(9.2.8) \quad \begin{array}{ccc} \mathit{cofiber} \left(\mathit{triv}_m^{nm}(\underline{\Lambda}_{\mathrm{BG}_{m,S}}(-1)[-2]) \xrightarrow{m \cdot c_1} \mathit{triv}_m^{nm}(\underline{\Lambda}_{\mathrm{BG}_{m,S}}) \right) & & \\ \downarrow m & & \downarrow 1 \\ \mathit{cofiber} \left(\underline{\Lambda}_{\mathrm{BG}_{m,S}}(-1)[-2] \xrightarrow{c_1} \underline{\Lambda}_{\mathrm{BG}_{m,S}} \right) & & \end{array}$$

It is then clear that, as Λ is torsion,

$$(9.2.9) \quad \mathit{triv}(\underline{\Lambda}_S) \simeq \varinjlim \left(\mathit{triv}_n^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{n \cdot \mu_\infty}) \rightarrow \mathit{triv}_{nm}^\infty((\Psi_{id,\eta}^t(\underline{\Lambda}_{[\mathbb{A}_S^1/\mathbb{G}_{m,S}]})^{nm \cdot \mu_\infty})) \right).$$

□

We will now need the following categorical lemma.

Lemma 9.3. *Let $\mathcal{C}_\bullet : \mathbb{N}_S^{\text{op}} \rightarrow \mathbf{Pr}_{\text{st}}^{\text{R}}$ be a diagram in the (big) ∞ -category of stable, presentable ∞ -categories with right adjoints. For every $n, m \in \mathbb{N}_S$, there is an adjunction*

$$(9.3.1) \quad i_{n,nm} : \mathcal{C}_n \rightleftarrows \mathcal{C}_{nm} : t_{n,nm}.$$

They are compatible in the obvious sense. Let $\mathcal{C} := \varprojlim_n \mathcal{C}_n$ be the limit of such diagram. For every $n \in \mathbb{N}_S$ we have an adjunction

$$(9.3.2) \quad i_n : \mathcal{C}_n \rightleftarrows \mathcal{C} : t_n.$$

These are also compatible in an obvious way.

For any $F \in \mathcal{C}$, let $F_n := t_n(F)$. As $t_{n,nm}(F_{nm}) \simeq F_n$, the counits of the adjunctions $(i_{n,nm}, t_{n,nm})$ induce morphisms

$$(9.3.3) \quad i_n(F_n) \rightarrow i_{nm}(F_{nm})$$

that define a diagram $\mathbb{N}_S \rightarrow \mathcal{C}$. The canonical morphism

$$(9.3.4) \quad \varinjlim_n i_n(F_n) \rightarrow F$$

is an equivalence.

Proof. Let $G \in \mathcal{C}$. Then

$$(9.3.5) \quad \text{Map}_{\mathcal{C}}(\varinjlim_n i_n(F_n), G) \simeq \varprojlim_n \text{Map}_{\mathcal{C}}(i_n(F_n), G) \simeq \varprojlim_n \text{Map}_{\mathcal{C}_n}(F_n, G_n),$$

where $G_n := t_n(G)$. $\text{Map}_{\mathcal{C}}(F, G)$ is the pullback of the following diagram

$$(9.3.6) \quad \begin{array}{ccc} & \text{Fun}(\Delta^1, \mathcal{C}) & \\ & \downarrow (ev_0, ev_1) & \\ \Delta^0 & \xrightarrow{(F, G)} & \mathcal{C} \times \mathcal{C}. \end{array}$$

As $\text{Fun}(\Delta^1, \mathcal{C}) \simeq \varprojlim_n \text{Fun}(\Delta^1, \mathcal{C}_n)$ and $\mathcal{C} \times \mathcal{C} \simeq \varprojlim_n \mathcal{C}_n \times \mathcal{C}_n$, this diagram is the limit of the diagrams

$$(9.3.7) \quad \begin{array}{ccc} & \text{Fun}(\Delta^1, \mathcal{C}_n) & \\ & \downarrow (ev_0, ev_1) & \\ \Delta^0 & \xrightarrow{(F_n, G_n)} & \mathcal{C}_n \times \mathcal{C}_n, \end{array}$$

whose pullbacks are $\text{Map}_{\mathcal{C}_n}(F_n, G_n)$. Since limits commute with limits, we find that

$$(9.3.8) \quad \text{Map}_{\mathcal{C}}(F, G) \simeq \varprojlim_n \text{Map}_{\mathcal{C}_n}(F_n, G_n).$$

The claim follows. □

In particular, we get that

Corollary 9.4. *The following equivalences hold in $\mathcal{D}_{\text{ét}}(\text{BG}_{m,S}; \Lambda)^{\mu_\infty}$:*

$$(9.4.1) \quad \text{triv}(\underline{\Delta}_{\text{BG}_{m,S}}) \simeq \varinjlim_n (\Psi_{id,\eta}^t)^{(n)}(\underline{\Delta}_S) \simeq \Psi_{id,\eta}^t(\underline{\Delta}_S).$$

Proposition 9.5. *With the same notation of the previous sections,*

$$(9.5.1) \quad (\text{triv}(\underline{\Lambda}_{X_0}))^{\mu_\infty} \simeq \text{cofiber}(\cdot: \underline{\Lambda}_{X_0}(-1)[-2] \xrightarrow{c_1(\mathcal{L}|_{X_0})} \underline{\Lambda}_{X_0}) \in \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

Proof. As $\underline{\Lambda}_{X_0} \simeq p_0^*(\underline{\Lambda}_{\text{BG}_{m,S}})$, we have that

$$(9.5.2) \quad (\text{triv}(\underline{\Lambda}_{X_0}))^{\mu_\infty} \simeq (\tilde{p}_0^*(\underline{\Lambda}_{\text{BG}_{m,S}}))^{\mu_\infty} \simeq p_0^*(\underline{\Lambda}_{\text{BG}_{m,S}})^{\mu_\infty}.$$

The previous corollary and the fact that $(-)^{\mu_\infty} \circ \Psi_{id,\eta}^t \simeq (\Psi_{id,\eta}^t)^{(1)} \simeq i_0^* \circ (j_0)_*$ thus imply that

$$(9.5.3) \quad (\text{triv}(\underline{\Lambda}_{X_0}))^{\mu_\infty} \simeq p_0^*(\underline{\Lambda}_{\text{BG}_{m,S}}(-1)[-2] \xrightarrow{c_1} \underline{\Lambda}_{\text{BG}_{m,S}}),$$

where $c_1 \in H_{\text{ét}}^2(\text{BG}_{m,S}, \underline{\Lambda}_{\text{BG}_{m,S}}(1))$ is the *universal first Chern class*. The claim follows as

$$(9.5.4) \quad p_0^*(c_1) = c_1(\mathcal{L}|_{X_0}) \in H_{\text{ét}}^2(X_0, \underline{\Lambda}_{X_0}(1)).$$

Indeed, we have the following commutative diagram:

$$(9.5.5) \quad \begin{array}{ccc} H_{\text{ét}}^1(X_0, \mathbb{G}_m) & \xrightarrow{\partial_{X_0}} & H_{\text{ét}}^2(X_0, \underline{\Lambda}(1)) \\ \uparrow p_0^* & & \uparrow p_0^* \\ H_{\text{ét}}^1(\text{BG}_{m,S}, \mathbb{G}_m) & \xrightarrow{\partial_{\text{BG}_{m,S}}} & H_{\text{ét}}^2(\text{BG}_{m,S}, \underline{\Lambda}(1)). \end{array}$$

Moreover,

$$(9.5.6) \quad \mathbb{Z} \subseteq H_{\text{ét}}^1(\text{BG}_{m,S}, \mathbb{G}_m)$$

The element $U \in H_{\text{ét}}^1(\text{BG}_{m,S}, \mathbb{G}_m)$ corresponding to 1 is the *universal $\mathbb{G}_{m,S}$ -torsor* and is such that

$$(9.5.7) \quad p_0^*(U) = [\mathcal{L}|_{X_0}], \quad \partial_{\text{BG}_{m,S}}(U) = c_1.$$

□

Theorem 9.6. *There is an equivalence*

$$(9.6.1) \quad (\Phi_p^t(\underline{\Lambda}_X))^{\mu_\infty} \simeq \Phi_{(X,S)}^{\text{mi}}(\underline{\Lambda}_X) \in \mathcal{D}_{\text{ét}}(X_0; \Lambda).$$

Proof. We have that

$$(9.6.2) \quad \Phi_{(X,S)}^{\text{mi}}(\underline{\Lambda}_X) \simeq \text{cofiber}(\text{cofiber}(c_1(\mathcal{L}|_{X_0}) : \underline{\Lambda}_{X_0}(-1)[-2] \rightarrow \underline{\Lambda}_{X_0}) \xrightarrow{f} i^* j_* \underline{\Lambda}_{U_X}),$$

where f is the morphism induced by the counit $\tilde{f} : \underline{\Lambda}_{X_0} \rightarrow i^* j_* \underline{\Lambda}_{U_X}$ ($\tilde{f} \circ c_1(\mathcal{L}|_{X_0}) \sim 0$).

On the other hand, the previous lemmata and propositions imply that

$$(9.6.3) \quad (\Phi_p^t(\underline{\Lambda}_X))^{\mu_\infty} \simeq \text{cofiber}(\text{cofiber}(c_1(\mathcal{L}|_{X_0}) : \underline{\Lambda}_{X_0}(-1)[-2] \rightarrow \underline{\Lambda}_{X_0}) \xrightarrow{sp^{\mu_\infty}} i^* j_* \underline{\Lambda}_{U_X}).$$

Therefore, we only need to show that the morphisms f and sp^{μ_∞} are homotopic. With the same notation as in Construction 2.11, we see that the composition

$$(9.6.4) \quad \underline{\Lambda}_{X_0} \rightarrow \text{cofiber}(c_1(\mathcal{L}|_{X_0}) : \underline{\Lambda}_{X_0}(-1)[-2] \rightarrow \underline{\Lambda}_{X_0}) \xrightarrow{sp^{\mu_\infty}} i^* j_* \underline{\Lambda}_{U_X}$$

is homotopic to \bar{sp} , i.e. to \tilde{f} . The claim follows. □

Remark 9.7. Using the usual limiting procedure, we find monodromy-invariant (\mathbb{Q}_ℓ -adic) vanishing cycles (see [25, Definition 4.2.6]) as

$$(9.7.1) \quad \Phi_{(X,s)}^{\text{mi}}(\mathbb{Q}_\ell) \simeq \Phi_p^t(\mathbb{Q}_{\ell,X})^{\mu_\infty}.$$

APPENDIX A.

Let \mathcal{A} be a filtered set with a minimal element e . We consider \mathcal{A} as an ordinary category by letting $\text{Hom}_{\mathcal{A}}(a, a')$ be the punctual set if $a \leq a'$ and the empty set otherwise. Let $X : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}$ be a diagram in a $(2, 1)$ -category. We assume that \mathcal{C} has 2-fibre products. Suppose that there is a distinguished class of 1-morphisms F in \mathcal{C} , which is stable under composition, pullbacks and contains all identities, and that $X(a \leq a') \in F$ for every $a \leq a' \in \mathcal{A}$. Moreover, suppose that we are given a 1-morphism in \mathcal{C}

$$(A.0.1) \quad i(e) : X_0(e) \rightarrow X(e).$$

We shall put $X_0(a) := X_0(e) \times_{X(e)} X(a)$ for all $a \in \mathcal{A}$. It follows from the properties of 2-fibre products that we get a diagram $X_0 : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}$, such that $X_0(a \leq a') \in F$ for every $a \leq a' \in \mathcal{A}$.

Starting from these data, we will construct a diagram

$$(A.0.2) \quad D : N(\mathcal{A}^{\text{op}}) \times \Delta^1 \rightarrow \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}.$$

Here N denotes the simplicial nerve of [20] and \mathcal{C} is regarded as a simplicial category by applying the nerve construction on all hom categories. The simplicial set $\delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}$ is the one defined in [19].

We will define D explicitly.

Remark A.1 (n -simplexes in $\delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}$). By definition, an n -simplex $\Delta^n \rightarrow \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}$ is a morphism of simplicial sets

$$(A.1.1) \quad \Delta^n \times (\Delta^n)^{\text{op}} \rightarrow N(\mathcal{C})$$

such that vertical morphisms lie in F and all squares are Cartesian. By definition of the simplicial nerve, this corresponds to simplicial functor

$$(A.1.2) \quad S : \mathbb{C}[\Delta^n \times (\Delta^n)^{\text{op}}] \rightarrow \mathcal{C}$$

such that $S((i, j) \leq (i', j))$ is in F for all $i \leq i'$ and all j and such that each diagram

$$(A.1.3) \quad \begin{array}{ccc} S(i, j) & \longrightarrow & S(i, j') \\ \downarrow & & \downarrow \\ S(i', j) & \longrightarrow & S(i', j') \end{array}$$

is a 2-fibre product, where $i \leq i', j \geq j'$.

Construction A.2. Let $n \geq 0$. We will now construct a function

$$(A.2.1) \quad D_n : \text{Hom}_{\text{Set}_\Delta}(\Delta^n, N(\mathcal{A}^{\text{op}}) \times \Delta^1) \rightarrow \text{Hom}_{\text{Set}_\Delta}(\Delta^n, \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}).$$

Let $(a, s) = (a_0 \leq a_1 \leq \dots \leq a_n, s_0 \leq \dots \leq s_n)$ be an n -simplex of $N(\mathcal{A}) \times \Delta^1$. We need to define a simplicial functor

$$(A.2.2) \quad D_n(a, s) : \mathbb{C}[\Delta^n \times (\Delta^n)^{\text{op}}] \rightarrow \mathcal{C}.$$

Let (i, j) be an object of $\mathbb{C}[\Delta^n \times (\Delta^n)^{op}]$. Then we put

$$(A.2.3) \quad D_n(a, s)(i, j) = \begin{cases} X(a_i) & \text{if } s_j = 1; \\ X_0(a_i) & \text{if } s_j = 0. \end{cases}$$

For each $(i, j), (i', j') \in \mathbb{C}[\Delta^n \times (\Delta^n)^{op}]$, we need to define a morphism of simplicial sets

$$(A.2.4) \quad \text{Map}_{\mathbb{C}[\Delta^n \times (\Delta^n)^{op}]}((i, j), (i', j')) = N(P_{(i,j),(i',j')}) \rightarrow N(\text{Hom}_{\mathcal{C}}(D_n(a, s)(i, j), D_n(a, s)(i', j'))),$$

that is, we have to define a functor

$$(A.2.5) \quad P_{(i,j),(i',j')} \rightarrow \text{Hom}_{\mathcal{C}}(D_n(a, s)(i, j), D_n(a, s)(i', j')).$$

Here $P_{(i,j),(i',j')}$ stands for the partially ordered set defined in [20, Example 1.1.5.9]. As a first approximation, notice that the morphism of diagrams $i : X_0 \rightarrow X$ provides us with a diagram

$$(A.2.6) \quad \begin{array}{cccccccccccccccc} X_0(a_n) & \xrightarrow{id} & X_0(a_n) & \xrightarrow{id} & \dots & \xrightarrow{id} & X_0(a_n) & \xrightarrow{i(a_n)} & X(a_n) & \xrightarrow{id} & X(a_n) & \xrightarrow{id} & \dots & \xrightarrow{id} & X(a_n) \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ X_0(a_{n-1}) & \xrightarrow{id} & X_0(a_{n-1}) & \xrightarrow{id} & \dots & \xrightarrow{id} & X_0(a_{n-1}) & \xrightarrow{i(a_{n-1})} & X(a_{n-1}) & \xrightarrow{id} & X(a_{n-1}) & \xrightarrow{id} & \dots & \xrightarrow{id} & X(a_{n-1}) \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & & & & \vdots \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ X_0(a_0) & \xrightarrow{id} & X_0(a_0) & \xrightarrow{id} & \dots & \xrightarrow{id} & X_0(a_0) & \xrightarrow{i(a_0)} & X(a_0) & \xrightarrow{id} & X(a_0) & \xrightarrow{id} & \dots & \xrightarrow{id} & X(a_0). \end{array}$$

The vertical maps are those defined by X_0 and X . The fact that condition (A.1.3) holds is obvious from the definitions. The 2-cells are not depicted. However, the 2-cells providing isomorphisms

$$(A.2.7) \quad X_0(a \leq a'') \simeq X_0(a' \leq a'') \circ X_0(a \leq a') \quad (\text{resp. } X(a \leq a'') \simeq X(a' \leq a'') \circ X(a \leq a'))$$

are provided by the functor X_0 (resp. X). Moreover, we attach the identity 2-cells in the squares that do not belong to the column where the $i(a_j)$ appear and 2-cells provided by the definition of 2-fibre products (see eg [27, Tag 003O]) to those that belong to that column. These are compatible in an obvious sense that we will not make explicit here (by the universal property of 2-fibre products).

Each $A \in P_{(i,j),(i',j')}$ admits a unique "minimal path", that is a "path" $(i, j) \rightarrow (i', j')$ such that there are no elements of A under this "path". At the level of objects, we define $P_{(i,j),(i',j')} \rightarrow \text{Hom}_{\mathcal{C}}(D_n(a, s)(i, j), D_n(a, s)(i', j'))$ by sending A to the composition of the images of the "1-steps" of this minimal path. If $A \subseteq A'$, then the minimal path of A' lies below that of A and we define the morphism

$$(A.2.8) \quad D_n(a, s)(A \subseteq A') : D_n(a, s)(A) \rightarrow D_n(a, s)(A')$$

by using the 2-cells mentioned above. The fact that

$$(A.2.9) \quad D_n(a, s)(A \subseteq A'') = D_n(a, s)(A' \subseteq A'') \circ D_n(a, s)(A \subseteq A')$$

follows from the compatibility of the 2-cells. The equality

$$(A.2.10) \quad D_n(a, s)(A \subseteq A) = id$$

is obvious.

This defines the functors $P_{(i,j),(i',j')} \rightarrow \text{Hom}_{\mathcal{C}}(D_n(a, s)(i, j), D_n(a, s)(i', j'))$, as desired.

Finally, notice that the square

$$(A.2.11) \quad \begin{array}{ccc} P_{(i,j),(i',j')} \times P_{(i',j'),(i'',j'')} & \longrightarrow & \text{Hom}_{\mathcal{C}}(D_n(a,s)(i,j), D_n(a,s)(i',j')) \times \text{Hom}_{\mathcal{C}}(D_n(a,s)(i',j'), D_n(a,s)(i'',j'')) \\ \downarrow \cup & & \downarrow \circ_e \\ P_{(i,j),(i'',j'')} & \longrightarrow & \text{Hom}_{\mathcal{C}}(D_n(a,s)(i,j), D_n(a,s)(i'',j'')) \end{array}$$

is commutative on the nose: at the level of objects, this follows immediately from the observation that if $A \in P_{(i,j),(i',j')}$ and $A' \in P_{(i',j'),(i'',j'')}$, then the "minimal path" of $A \cup A'$ is the join of the minimal paths of A and A' . At the level of morphisms, it follows once again from the compatibility of the 2-cells in the diagram above. This provides us with the desired simplicial functor $D_n(a,s) : \mathbb{C}[\Delta^n \times (\Delta^n)^{op}] \rightarrow \mathcal{C}$.

Lemma A.3. *The functions $D_n : \text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, N(\mathcal{A}^{op}) \times \Delta^1) \rightarrow \text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}})$ assemble in a morphism of simplicial sets*

$$(A.3.1) \quad D : N(\mathcal{A}^{op}) \times \Delta^1 \rightarrow \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}$$

Proof. We need to show that the D_n 's are compatible with the face and degeneracy maps of $N(\mathcal{A}^{op}) \times \Delta^1$ and $\delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}$. The description of face and degeneracy maps in $N(\mathcal{A}^{op}) \times \Delta^1$ being well known, we will limit to describe those of the $\delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}$. For $S : \mathbb{C}[\Delta^n \times (\Delta^n)^{op}] \rightarrow N(\mathcal{C})$, its image along the degeneracy map

$$(A.3.2) \quad s_n^i : \text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}}) \rightarrow \text{Hom}_{\text{Set}_{\Delta}}(\Delta^{n+1}, \delta_{2,\{2\}}^*(N(\mathcal{C}))_{F,\text{all}}^{\text{cart}})$$

corresponds to inserting a row

$$(A.3.3) \quad \begin{array}{ccccccc} S(i,n) & \longrightarrow & S(i,n-1) & \longrightarrow & \cdots & \longrightarrow & S(i,1) & \longrightarrow & S(i,0) \\ \downarrow id & & \downarrow id & & & & \downarrow id & & \downarrow id \\ S(i,n) & \longrightarrow & S(i,n-1) & \longrightarrow & \cdots & \longrightarrow & S(i,1) & \longrightarrow & S(i,0), \end{array}$$

a column

$$(A.3.4) \quad \begin{array}{ccc} S(0,i) & \xrightarrow{id} & S(0,i) \\ \downarrow & & \downarrow \\ S(1,i) & \xrightarrow{id} & S(1,i) \\ \downarrow & & \downarrow \\ \cdots & & \cdots \\ \downarrow & & \downarrow \\ S(n-1,i) & \xrightarrow{id} & S(n-1,i) \\ \downarrow & & \downarrow \\ S(n-1,0) & \xrightarrow{id} & S(n,i), \end{array}$$

and the obvious 2-cells. It is then clear from this description and that of the $D_n(a_0 \leq \cdots \leq a_i \leq a_i \leq \cdots \leq a_n, s_0 \leq \cdots \leq s_i \leq s_i \leq \cdots \leq s_n)$ that the D_n 's are compatible with the degeneracy maps.

On the other hand, the face maps

$$(A.3.5) \quad d_n^i : \text{Hom}_{\text{Set}_\Delta}(\Delta^n, \delta_{2, \{2\}}^*(N(\mathcal{C}))_{F, \text{all}}^{\text{cart}}) \rightarrow \text{Hom}_{\text{Set}_\Delta}(\Delta^{n-1}, \delta_{2, \{2\}}^*(N(\mathcal{C}))_{F, \text{all}}^{\text{cart}})$$

correspond to taking compositions $S(k+1, i) \circ S(k, i)$ and $S(i, j-1) \circ S(i, j)$ to get rid of the i^{th} row and column in the diagram $S : \Delta^n \times (\Delta^n)^{\text{op}} \rightarrow \mathcal{C}$. Then it is also obvious that the D_n 's are compatible with the face maps as well. \square

Corollary A.4. *The above construction provides us with an ∞ -functor*

$$(A.4.1) \quad N(\mathcal{A}^{\text{op}}) \times \Delta^1 \rightarrow \text{Corr}(N(\mathcal{C}))_{F, \text{all}}.$$

Proof. This follows immediately from [19, Section 6.1]. \square

REFERENCES

- [1] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *American Journal of Mathematics*, 130(5):1337–1398, 2008.
- [2] M. Artin and P. Deligne. *Théorie des Topos et Cohomologie Étale des Schémas*, volume 305 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1973. dirigé par M. Artin, A. Grothendieck and J.-L. Verdier.
- [3] J. Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I). *Astérisque*, 314:x+464, 2007.
- [4] J. Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (II). *Astérisque*, 315:vi+362, 2007.
- [5] J. Ayoub. La réalisation étale et les opérations de Grothendieck. *Annales scientifiques de l'ENS*, 47(1):1–145, 2014.
- [6] Kai A. Behrend. Derived ℓ -adic categories for algebraic stack. *Memoirs of the American Mathematical Society*, 163(774):viii+93, 2003.
- [7] C. Cadman. Using stacks to impose tangency conditions on curves. *American Journal of Mathematics*, 129(2):405–427, 2007.
- [8] D.-C. Cisinski and F. Déglise. Étale motives. *Compositio Mathematica*, 152(3):556–666, 2016.
- [9] D.-C. Cisinski and F. Déglise. *Triangulated Categories of Mixed Motives*. Springer Monographs in Mathematics. Springer Cham, 2019.
- [10] P. Deligne, A. Grothendieck, Michèle Raynaud, M. Raynaud, and D. S. Rim. *Groupes de Monodromie en Géométrie Algébrique*, volume 288 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1972. dirigé par A. Grothendieck avec la collaboration de M. Raynaud et D. S. Rim.
- [11] P. Deligne and N. Katz. *Groupes de Monodromie en Géométrie Algébrique*, volume 340 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1973.
- [12] D. Gaitsgory and N. Rozenblyum. *A Study in Derived Algebraic Geometry: Volume I: Correspondences and Duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.
- [13] A. Grothendieck, B. Saint-Donat, and J. L. Verdier. *Théorie des Topos et Cohomologie Étale des Schémas*, volume 270 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1972. dirigé par M. Artin, A. Grothendieck and J.-L. Verdier.
- [14] A. Grothendieck and J. L. Verdier. *Théorie des Topos et Cohomologie Étale des Schémas*, volume 269 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1972. dirigé par M. Artin, A. Grothendieck and J.-L. Verdier.
- [15] F. Jin and E. Yang. Some results on the motivic nearby cycle. arXiv:2107.08603v4, 2021. preprint.
- [16] A. Khan. Virtual fundamental classes of derived stacks I. arXiv:1909.01332v1, 2019. preprint.
- [17] Y. Laszlo and M. Olsson. The six operations for sheaves on Artin stacks I: Finite coefficients. *Publ. Math. IHES*, 107:109–168, 2008.
- [18] G. Laumon and L. Moret-Bailly. *Champs algébriques*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer Berlin, Heidelberg, 2000.

- [19] Y. Liu and W. Zheng. Enhanced six operations and base change theorem for higher Artin stacks. arXiv:1211.5948v3. preprint.
- [20] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [21] J. Lurie. Higher algebra, 2017. available on the author's webpage.
- [22] J. Milnor. *Singular points of complex hypersurfaces*, volume 61 of *Annals of Mathematics Studies*. Princeton University Press and University of Tokyo Press, Princeton, NJ and Tokyo, 1968.
- [23] M. Olsson. Sheaves on artin stacks. *J. Reine Angew. Math.*, 2007(603):55–112, 2007.
- [24] M. Pippi. *Catégories de singularités, factorisations matricielles et cycles évanescents*. PhD thesis, Université Toulouse III - Paul Sabatier, 2020.
- [25] M. Pippi. Motivic and ℓ -adic realizations of the category of singularities of the zero locus of a global section of a vector bundle. *Sel. Math. New Ser.*, 28(33), 2022.
- [26] J. P. Serre. *Corps Locaux*. Actualités Scientifiques et Industrielles 1296. Publications de l'Institut de Mathématique de l'Université de Nancago. 8, Paris: Hermann and Cie, 1962.
- [27] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>, 2022.
- [28] J.-L. Verdier. Spécialisation de faisceaux et monodromie modérée. In *Analyse et topologie sur les espaces singuliers (II-III) - 6–10 juillet 1981*, number 101–102 in Astérisque. Société mathématique de France, 1983.

Email address: denis-charles.cisinski@ur.de

FAKULTÄT FÜR MATHEMATIK UNIVERSITÄT REGENSBURG 93040 REGENSBURG GERMANY

Email address: massimo.pippi@ur.de

FAKULTÄT FÜR MATHEMATIK UNIVERSITÄT REGENSBURG 93040 REGENSBURG GERMANY