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Recollection about last week

Def: $\forall S$ profinite set $\mathbb{Z}[S]^{\blacksquare} := \lim\limits_{\substack{S \twoheadrightarrow S' \\ S' \text{ finite set}}} \mathbb{Z}[S']$.

Remark: The limit is derived

$$\mathbb{Z}[S]^{\blacksquare} = R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z}), \mathbb{Z}) \simeq R\underline{\mathrm{Hom}}(\mathcal{C}(S,\mathbb{Z}), \mathbb{Z})$$

Def: $\cdot$ $M \in \mathrm{CondAb}$ is solid

$$\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, M) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[S], M)$$

$\cdot$ $X \in \mathcal{D}(\mathrm{CondAb})$ is (derived) solid

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, X) \xrightarrow{\sim} R\mathrm{Hom}(\mathbb{Z}[S], X)$$

① S profinite $\quad \mathcal{C}(S,\mathbb{Z}) \simeq \bigoplus\limits_I \mathbb{Z} \Rightarrow \mathbb{Z}[S]^{\blacksquare} \simeq \prod\limits_I \mathbb{Z}$

② $\mathbb{Z}[S]^{\blacksquare}$ is (derived) solid.

③ $\mathrm{Solid} \subseteq \mathrm{CondAb}$, $\underline{\mathcal{D}(\mathrm{Solid}) \to \mathcal{D}(\mathrm{CondAb})}$ are reflective abelian/stable subcategories

Def: $\mathcal{C}$ stable ∞-cat, eq. w/ t-structure. $X \in \mathcal{C}$ is pseudocoherent (almost perfect) if $\mathrm{map}_{\mathcal{C}}(X, -) : \mathcal{C}_{\leq 0} \to \mathrm{Sp}$ commutes w/ filtered colimits

Thm: $\mathcal{C} = \mathcal{D}(\mathcal{A})$ $\mathcal{A}$ Grothendieck ab. cat w/ a family of compact proj. gen. $\{P_i\}$

$\Rightarrow$ $X \in \mathcal{C}$ is pseudocoherent iff it can be represented by a bounded below complex whose terms are finite sums of P_i's

Remark X pseudocoherent, $\{A_i\} \in \mathcal{A}$ family

$$\mathrm{map}_{\mathcal{C}}(X, \bigoplus_I A_i) \simeq \bigoplus_I \mathrm{map}_{\mathcal{C}}(X, A_i)$$

Lemma: X compact Hausdorff space, $\mathbb{Z}[X] \in \mathcal{D}(\mathrm{Cond}\,Ab)$ is pseudocoherent

Proof: $S \twoheadrightarrow X$ S extremely disconnected

$\Rightarrow \mathbb{Z}[X]$ has a resolution

ext. discon. compact proj

$$\cdots \to \mathbb{Z}[S \times_X S] \to \mathbb{Z}[S] \to \mathbb{Z}[X] \to 0 \qquad . \square$$

Lemma: • A compact ab. group $\Rightarrow A$ pseudocoherent

• $\mathbb{R}$ pseudocoherent

Proof: Two times ago we've seen $\exists$ a resolution

$$\cdots \to \bigoplus \mathbb{Z}[A^{n_i}] \to \mathbb{Z}[A] \to A \to 0$$

By the previous lemma all terms are pseudocoherent $\Rightarrow A$ pseudocoherent.

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0 \qquad \square$$

pseudocoherent $\Rightarrow \mathbb{R}$ is too.

Thm: Let $C \in \mathcal{D}(\mathrm{Cond}\,Ab)$ represented by a complex of the form

degree 0 0 forever

$$\cdots \to \bigoplus_{j \in J_1} \prod_{k \in K_{1,j}} \mathbb{Z} \to \bigoplus_{j \in J_0} \prod_{k \in K_{0,j}} \mathbb{Z} \to 0 \to 0 \to \cdots$$

Then $\forall S$ profinite set

internal hom

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$$

is an equivalence

We'll call such complexes "good".

Prop: Let C be a good complex. Then the objects

$$R\underline{\mathrm{Hom}}\Big(\prod_I \mathbb{R}/\mathbb{Z}, C\Big) \quad , \quad R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$$

are (-1)-connective

Prop: Let C be a good complex. Then the objects

$$\underline{R\mathrm{Hom}}\Big(\prod_I \mathbb{R}/\mathbb{Z}, C\Big) \quad , \quad \underline{R\mathrm{Hom}}(\mathbb{Z}[S], C)$$

are (-1)-connective

Assuming the proposition, let us prove the theorem.

Step 1: $\underline{R\mathrm{Hom}}(\mathbb{R}, C) = 0$

First assume $C = \bigoplus_{j\in J} \prod_{K_j} \mathbb{Z}$ in degree 0.

$$\underline{R\mathrm{Hom}}(\mathbb{R}, C) \overset{\text{pseudocoherence}}{\simeq} \bigoplus_{j\in J} \prod_{K_j} \underbrace{\underline{R\mathrm{Hom}}(\mathbb{R}, \mathbb{Z})}_{\text{two talks ago}} = 0$$

For any C we can take filtration "bête"

$$\underset{[\cdots 0 \to C_n \to \cdots \to C_0]}{W_{\leq n} C} \to C \to W_{\geq n+1} C \qquad \underline{R\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \underset{n\text{-connective (prop.)}}{W_{\geq n+1} C})$$

$$\underbrace{\underline{R\mathrm{Hom}}(\mathbb{R}, W_{\leq n} C)}_{0 \text{ by induction on } n} \to \underline{R\mathrm{Hom}}(\mathbb{R}, C) \to \underbrace{\underline{R\mathrm{Hom}}(\mathbb{R}, W_{\geq n+1} C)}_{\underline{R\mathrm{Hom}}(\mathbb{R}, W_{\geq n+1} C[-n-1])[n+1]} \to W_{\geq n+1} C$$

$\Rightarrow$ it is n-connective

$\Rightarrow \underline{R\mathrm{Hom}}(\mathbb{R}, C)$ is n-connective, but n arbitrary

$\Rightarrow \underline{R\mathrm{Hom}}(\mathbb{R}, C) = 0$.

Step 2: $\underline{R\mathrm{Hom}}\Big(\prod_I \mathbb{R}, C\Big) = 0$

$$\| \qquad\qquad \overset{0}{\|}$$

$$\underline{R\mathrm{Hom}}_{\mathbb{R}}\Big(\prod_I \mathbb{R}, \underline{R\mathrm{Hom}}(\mathbb{R}, C)\Big) \quad . \quad \square$$

Step 3: $\underline{R\mathrm{Hom}}(\mathbb{Z}[S]^{*}, C)$ is (-1)-connective

$$\|$$

$$\underline{R\mathrm{Hom}}\Big(\prod_I \mathbb{Z}, C\Big)$$

$$0 \to \prod_I \mathbb{Z} \to \prod_I \mathbb{R} \to \prod_I \mathbb{R}/\mathbb{Z} \to 0 \qquad (AB_4{}^*)$$

Step 3: $R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\circledast}, C)$ is (-1)-connective

$$\shortparallel$$

$$R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{Z}, C\right)$$

$$0 \to \prod_I \mathbb{Z} \to \prod_I \mathbb{R} \to \prod_I \mathbb{R}/\mathbb{Z} \to 0 \qquad (AB_6{}^*)$$

$$\underbrace{R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{R}/\mathbb{Z}, C\right)}_{(-1)\text{-connective.}} \to \underbrace{R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{R}, C\right)}_{=0} \to \underbrace{R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{Z}, C\right)}_{\text{connective ?}}$$

Step 4: Profit!

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\circledast}, C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$$

Let's do it first in degree 0, $C = \bigoplus_J \prod_{K_i} \mathbb{Z}$

The RHS, since $\mathbb{Z}[S]$ is pseudocoherent, is just $\bigoplus_J \prod_{K_i} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z})$

$\mathbb{Z}[S]^{\circledast} \simeq \prod_I \mathbb{Z}$ comp. ab. group $\Rightarrow$ pseudocoh.

$\prod_I \mathbb{Z} \to \prod_I \mathbb{R} \to \prod_I \mathbb{R}/\mathbb{Z}$ $\quad R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{Z}, \bigoplus_J \prod_{K_i} \mathbb{Z}\right) \simeq \bigoplus_J \prod_{K_i} R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{Z}, \mathbb{Z}\right)$

because the other terms of the SES do.

$\Rightarrow R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\circledast}, \mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z})$ (essentially prop. 4.3. in Nuno's talk)

$\Rightarrow$ By induction the thesis is true for $W_{\le n} C$ $\forall n$.

$$W_{\le n} C \to C \to W_{\ge n+1} C$$

the fiber is $(n-)$-connective

Step 3 / n-con

proposition

$$\begin{array}{ccccc} R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\circledast}, W_{\le n} C) & \to & R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\circledast}, C) & \to & R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\circledast}, W_{\ge n+1} C) \\ ?\downarrow & & \downarrow & & \downarrow \\ R\underline{\mathrm{Hom}}(\mathbb{Z}[S], W_{\le n} C) & \to & R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C) & \to & R\underline{\mathrm{Hom}}(\mathbb{Z}[S], W_{\ge n+1} C) \end{array}$$

$\Rightarrow$ the fiber of the central map is n-conn. $\forall n \Rightarrow$ it's 0.

Prop: Let C be a good complex. Then the objects

$$R\underline{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, C) \quad , \quad R\underline{Hom}(\mathbb{Z}[S], C)$$

are (-1)-connective

Proof: We fit C in a SES of complexes

$$C \to C_{\mathbb{R}} \to C_{\mathbb{R}/\mathbb{Z}}$$

$$C_n = \bigoplus_{j \in J_n} \prod_{k \in K_{n,j}} \mathbb{Z} \quad , (C_{\mathbb{R}})_n = \bigoplus_{j \in J_n} \prod_{k \in K_{n,j}} \mathbb{R} \quad , \quad C_{\mathbb{R}/\mathbb{Z}} = \bigoplus_{j \in J_n} \prod_{k \in K_{n,j}} \mathbb{R}/\mathbb{Z}$$

$\exists C_{\mathbb{R}}$ you have to show $R\underline{Hom}(\prod \mathbb{R}, \mathbb{R}) \simeq R\underline{Hom}(\prod \mathbb{Z}, \mathbb{R})$

$$\begin{array}{ccc} C_n & \to & C_{n-1} \\ \downarrow & & \downarrow \\ C_{\mathbb{R},n} & \dashrightarrow & C_{\mathbb{R},n-1} \end{array}$$

The fiber of that map $R\underline{Hom}(\prod \mathbb{R}/\mathbb{Z}, \mathbb{R}) = 0$.

Prove: $R\underline{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, C_{\mathbb{R}})$, $R\underline{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, C_{\mathbb{R}/\mathbb{Z}})$

$R\underline{Hom}(\mathbb{Z}[S], C_{\mathbb{R}})$, $R\underline{Hom}(\mathbb{Z}[S], C_{\mathbb{R}/\mathbb{Z}})$

are connective

Step 1: C 0-dimensional $C_{\mathbb{R}}$

$$R\underline{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, \bigoplus_J \prod_{K_j} \mathbb{R}) \simeq \bigoplus_J \prod_{K_j} R\underline{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, \mathbb{R}) = 0$$

$$\text{"} \quad \text{"} \quad (\quad \mathbb{R}/\mathbb{Z})^{\vee} = \bigoplus_I \mathbb{Z}$$

$R\underline{Hom}(\mathbb{Z}[S], \mathbb{R})$ conc. in degree 0.

Step 2: C bounded ✓

$\Rightarrow$ The thesis is true for $W_{\le n} C$ $\forall n$.

$$W_{\le n-1} C_{(\mathbb{R})/(\mathbb{R}/\mathbb{Z})} \longrightarrow \tau_{\le n} C_{(\mathbb{R})/(\mathbb{R}/\mathbb{Z})} \longrightarrow \operatorname{coker} d_{n+1} [n]$$

the comm. ext. are true here

connective if $n \gg 0$

$$R\underline{\mathrm{Hom}}(-, W_{\le n-1} C) \to \quad \to R\underline{\mathrm{Hom}}(-, \operatorname{coker} d_{n+1})[n]$$

$\Rightarrow$ It's enough to prove a comm. ext. for $\operatorname{coker} d_n$ $\forall n$

$$R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{R}/\mathbb{Z}, \operatorname{coker} d_n^{\mathbb{R}, \mathbb{R}/\mathbb{Z}}\right), \quad R\underline{\mathrm{Hom}}\left(\mathbb{Z}[S], \operatorname{coker} d_n^{\mathbb{R}, \mathbb{R}/\mathbb{Z}}\right)$$

are -2-connective

$$\begin{array}{ccccccccccc}
0 & \to & 0 & \to & C_{n-1} & \to & C_{n-2} & \to & \cdots & \to C_0 \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
0 & \to & \operatorname{coker} d_{n+1} & \to & C_{n-1} & \to & C_{n-2} & \to & \cdots &
\end{array}$$

$$\operatorname{coker} d_n = \operatorname{coker}\left(\bigoplus_S \prod_K \mathbb{Z} \longrightarrow \bigoplus_{S'} \prod_{K'} \mathbb{Z}\right)$$

by pseudocoherence we can assume S, S' finite $\Rightarrow$ we can assume

$$d_n : \prod_K \mathbb{Z} \longrightarrow \prod_{K'} \mathbb{Z}$$

$$\operatorname{coker} d_n^{\mathbb{R}/\mathbb{Z}} : \prod_K \mathbb{R}/\mathbb{Z} \longrightarrow \prod_{K'} \mathbb{R}/\mathbb{Z} \quad \text{is a compact abelian group}$$

$\Rightarrow$ has hom dim ≤ 1 $\quad R\underline{\mathrm{Hom}}\left(\prod_I \mathbb{R}/\mathbb{Z}, \operatorname{coker} d_n^{\mathbb{R}/\mathbb{Z}}\right)$ is in degrees $0, -1$.

$$d_n^{\mathbb{R}} : \prod_K \mathbb{R} \longrightarrow \prod_{K'} \mathbb{R}$$

(double dual of $\bigoplus \mathbb{Z}$ is $\bigoplus \mathbb{Z}$)

$$\prod_K \mathbb{Z} \longrightarrow \prod_{K'} \mathbb{Z} \quad \text{is the dual of a map} \quad \bigoplus_{K'} \mathbb{Z} \longrightarrow \bigoplus_K \mathbb{Z}$$

$$d_n^{\mathbb{R}} : \underline{\mathrm{Hom}}_{\mathbb{R}}\left(\bigoplus_K \mathbb{R}, \mathbb{R}\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{R}}\left(\bigoplus_{K'} \mathbb{R}, \mathbb{R}\right)$$

$$d_n^{\mathbb{R}} : \prod_K \mathbb{R} \to \prod_{K'} \mathbb{R}$$

(double dual of $\bigoplus \mathbb{Z}$ is $\bigoplus \mathbb{Z}$)

$\prod_K \mathbb{Z} \to \prod_{K'} \mathbb{Z}$ is the dual of a map $\bigoplus_{K'} \mathbb{Z} \to \bigoplus_K \mathbb{Z}$

$$d_n^{\mathbb{R}} : \underline{\mathrm{Hom}}_{\mathbb{R}}\left(\bigoplus_K \mathbb{R}, \mathbb{R}\right) \to \underline{\mathrm{Hom}}_{\mathbb{R}}\left(\bigoplus_{K'} \mathbb{R}, \mathbb{R}\right)$$

$R\underline{\mathrm{Hom}}(-, \mathbb{Z})$

$\underline{\mathrm{Hom}}(-, \mathbb{Z})$

$\bigoplus_K \mathbb{R} \to \bigoplus_{K'} \mathbb{R}$ map of $\mathbb{R}$-v. spaces (split inj followed by split surj)

$\Rightarrow \operatorname{coker} d_n^{\mathbb{R}} \simeq \prod_{\tilde{K}} \mathbb{R}$

the corr. ext. are computations we already did. $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{R}) \overset{\text{"real condensed cohomology"}}{=} \mathcal{C}(S, \mathbb{R})$.

$$R\underline{\mathrm{Hom}}(\sim, C_{\mathbb{R}, \mathbb{R}/\mathbb{Z}}) = \lim_n R\underline{\mathrm{Hom}}(\sim, \tau_{\leq n} C_{\mathbb{R}, \mathbb{R}/\mathbb{Z}})$$

(-1)-connective

$$d : \bigoplus_I \prod \mathbb{Z} \to \bigoplus \prod \mathbb{Z}$$

$$\operatorname{colim}_{\substack{\# I_0 < \infty \\ \# I_0' < \infty}} \bigoplus_{I_0} \prod \mathbb{Z} \to \bigoplus_{I_0'} \prod \mathbb{Z}$$

Thm: (i) Solid $\subseteq$ CondAb is closed under limits, colimits, extensions

Moreover $\exists$ a left adj $(-)^{\blacksquare} : \mathrm{CondAb} \to \mathrm{Solid}$ sending $\mathbb{Z}[S]$ to $\mathbb{Z}[S]^{\blacksquare}$.

(ii) The compact projective objects of Solid are precisely $\prod_I \mathbb{Z}$.

(iii) The functor $\mathcal{D}(\mathrm{Solid}) \to \mathcal{D}(\mathrm{Cond\,Ab})$ is fully faithful, the image is closed under limits, colimits & extensions, is spanned by those complexes w/ solid homology, has a left adj which is the der. functor of $(-)^{\blacksquare}$.

Thm: (i) Solid $\subseteq$ Cond Ab is closed under limits, colimits, extensions

Moreover $\exists$ a left adj $(-)^{■}$: Cond Ab $\to$ Solid sending $\mathbb{Z}[S]$ to $\mathbb{Z}[S]^{■}$.

(ii) The compact projective objects of Solid are precisely $\prod_I \mathbb{Z}$.

(iii) The functor $\mathcal{D}(\text{Solid}) \to \mathcal{D}(\text{Cond Ab})$ is fully faithful, the image is closed under limits, colimits & extensions, is spanned by those complexes w/ solid homology, has a left adj which is the der. functor of $(-)^{■}$.

(iv) $\mathcal{D}(\text{Solid})$ is "compactly generated" $\mathcal{D}(\text{Solid}) = \text{Ind}\, \mathcal{D}(\text{Solid})^{\omega}$

(v) $(\mathcal{D}(\text{Solid})^{\omega})^{op} \to \mathcal{D}^b(\mathbb{Z})$ is an eq.

$M \longmapsto R\underline{Hom}(M, \mathbb{Z})$

(vi) $R\underline{Hom}(\mathbb{Z}[S]^{■}, C) \xrightarrow{\sim} R\underline{Hom}(\mathbb{Z}[S], C)$ $\forall C \in \mathcal{D}(\text{Solid})$

(vii) $\mathbb{R}^{L■} = 0$

Proof: (i), (iii) follow formally

(ii) $\mathbb{Z}[S]^{■}$ are compact project. generators $\Rightarrow$ every comp. proj object is a retract of one of $\mathbb{Z}[S]^{■} \simeq \prod_I \mathbb{Z}$ but we can use the double dual isl as before

$R\underline{Hom}(\prod_I \mathbb{Z}, \mathbb{Z}) \simeq \bigoplus_I \mathbb{Z}$

$\Rightarrow$ it's enough to prove that a summand of a free $\mathbb{Z}$-module is free (comm. alg. exercise).

(iv) size issues, read carefully.

(v) $R\underline{Hom} : \mathcal{D}(\text{Solid})^{\omega} \to \mathcal{D}^b(\mathbb{Z})^{op}$

$\mathcal{D}(\text{Solid})^{\omega}$ is gen. by $\prod_I \mathbb{Z}$ under finite colimits

$R\underline{Hom} : \mathcal{D}^b(\mathbb{Z})^{op} \to \mathcal{D}(\text{Solid})^{\omega}$

$\bigoplus_I \mathbb{Z} \longmapsto \prod_I \mathbb{Z}$

The comp. is eq. to the identity on compact proj. generators,

(vi) works w/ truncating

(vii) $\mathbb{R}^{L\blacksquare} = 0$

$R\underline{Hom}(\mathbb{R}, C) = 0$ for connective objects of $\mathcal{D}(\text{Solid})$

& conn. ext. conclude. $\square$

<u>Remark</u>: From pt (vi) $(M \otimes^L N)^{\blacksquare} \to (M^{\blacksquare} \otimes^L N)^{\blacksquare}$ is an eq.

$$R\underline{Hom}(N, R\underline{Hom}(M, P)) \xleftarrow{\sim} R\underline{Hom}(N, R\underline{Hom}(M^{\blacksquare}, P))$$

$$R\underline{Hom}(M^{\blacksquare}, P) \to R\underline{Hom}(M, P)$$

but we can prove it for $\mathbb{Z}[S]$ gen. of $\mathcal{D}(\text{CondAb})$ and that was (vi).

$\Rightarrow$ $\text{CondAb} \to \text{Solid}$, $\mathcal{D}(\text{CondAb}) \to \mathcal{D}(\text{Solid})$ are comp. w/ the symm. monoidal structure.

<u>Prop</u>: $\prod_I \mathbb{Z} \otimes^{L\blacksquare} \prod_J \mathbb{Z} \simeq \prod_{I \times J} \mathbb{Z}$

<u>Proof</u>: Up to reduction it's enough to prove $\mathbb{Z}[S]^{\blacksquare} \otimes^{L\blacksquare} \mathbb{Z}[T]^{\blacksquare} \simeq \mathbb{Z}[S \times T]^{\blacksquare}$
& that is exactly the symm. monoid. of $(-)^{\blacksquare}$. $\square$

<u>Cor</u>: $\mathbb{Z}[[U]] \otimes^{L\blacksquare} \mathbb{Z}[[T]] \simeq \mathbb{Z}[[U,T]]$ $(p \neq \ell)$

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_\ell \simeq 0$

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_p$

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{R} \simeq 0$

$\mathbb{Z}_p \otimes^{\blacksquare} \mathbb{Z}[[U]] \simeq \mathbb{Z}_p[[U]]$

$$\mathbb{Z}[[U]] = \lim_n \mathbb{Z}[U]/_{U^n} \simeq \lim_n \bigoplus_{i=0}^{n} \mathbb{Z} = \lim_n \prod_{i=0}^{n} \mathbb{Z}$$

<u>Proof</u>: The first is the lemma $\mathbb{Z}[[U]] \simeq \prod_{n \geq 0} \mathbb{Z}$

The others follow from $\mathbb{R}^{L\blacksquare} = 0$, & the SES.

$$0 \to \mathbb{Z}[[U]] \xrightarrow{U-p} \mathbb{Z}[[U]] \to \mathbb{Z}_p \to 0. \qquad \square$$

Prop: X cw complex $\mathbb{Z}[X]^{L\blacksquare} \simeq C_*(X;\mathbb{Z})$

Proof: Suppose X finite, $\mathbb{Z}[X]^{L\blacksquare}$ pseudocoherent in $\mathcal{D}(\text{Solid})$

$C_*(X,\mathbb{Z})$ pseudocoh.

$R\underline{\mathrm{Hom}}(-,\mathbb{Z})$ is fully faith. on pseudocoherents

$R\underline{\mathrm{Hom}}(\mathbb{Z}[X]^{L\blacksquare},\mathbb{Z}) \simeq C^*_{cond}(X,\mathbb{Z})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$R\underline{\mathrm{Hom}}(C_*(X,\mathbb{Z}),\mathbb{Z}) \simeq C^*(X,\mathbb{Z})$. $\square$