8. Solid A-modules

Debt from last time

Prop 7.14: $\mathcal{A}, \mathcal{B}$ analytic, $f : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$ map of cond rings, $\mathcal{A}[S] \to \mathcal{B}[S]$ a nat transformation. If for all EDS $S \to \underline{\mathcal{A}}$, the square

$$\begin{tikzcd} \mathcal{A}[S] \arrow[r] \arrow[d] & \mathcal{A}[pt] \arrow[d] \\ \mathcal{B}[S] \arrow[r] & \mathcal{B}[pt] \end{tikzcd}$$

commutes, f is a map of analytic rings.

Question: Why is $\mathbb{Z}_\blacksquare[S] \overset{L}{\otimes}_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathbb{Z}_{p,\blacksquare}[S]$?

Well actually the proof claimed that
$\mathbb{Z}_\blacksquare[S] \overset{L}{\otimes}_{\mathbb{Z}_\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_{p,\blacksquare}[S]$ which is easy

Every discrete ring f.g. $/\mathbb{Z}$

Recall:

1) $A_\blacksquare$: $S \in \text{EDS}$, $A_\blacksquare[S] := \varprojlim_i A[S_i]$

$$\simeq \underline{\mathrm{Hom}}_{\mathbb{Z}}(C(S,\mathbb{Z}), A)$$

$\simeq$ —

$\simeq \prod_I A$

2) $R \to A$: $(A,R)_\blacksquare [S] := A \overset{L}{\otimes}_R R_\blacksquare [S]$

Last time: $R_\blacksquare$ analytic
$\Rightarrow$ so is $(A,R)_\blacksquare$

Goals: • $A_\blacksquare$ is analytic
• Study pushforwards w/ compact supp $f_!$
& exc. inv. image $f^!$ & related functors
(Affine Coherent duality

Thm:
(i) $(A,R)_\blacksquare$ is analytic
(ii) $R \to S \to A$

$j^*: D((A,R)_\blacksquare) \rightleftarrows D((A,S)_\blacksquare): j_*$ forget

j^* admits a left adj.
$j_!: D((A,S)_\blacksquare) \hookrightarrow D((A,R)_\blacksquare)$

& explicit formula
(iii) $f: \mathrm{Spec}(A) \to \mathrm{Spec}(R)$

forget

$$f_! : D(A_\blacksquare) \xrightarrow{\sim} D((A,R)_\blacksquare) \longrightarrow D(R_\blacksquare)$$

$$A_\blacksquare = (A,A)_\blacksquare$$

$f_!$ preserves colims, satisfies proj. formula

+ nice things

(IV) $f_! \dashv f^!$

+ nice things

Pf strategy: $R \to A$ can factored

$$R \rightsquigarrow R[x_1, \dots, x_n] \twoheadrightarrow A$$

main work $\qquad\qquad$ easy

We will concentrate on case $R \to R[x]$

Note: $R \to A$ finite $\Rightarrow$ $(A,R)_\blacksquare = A_\blacksquare$

$$(A,R)_\blacksquare[S] = A \otimes_R \prod_I R = \prod_I A$$

The case of $\mathbb{A}^1$

R (finite hom. dim.) $R_\blacksquare$ analytic ($R = \mathbb{Z}$)

$A = R[t]$

<u>Thm</u> $A_\blacksquare$ is analytic

<u>Pf</u>: Need: $C_* = \cdots \to C_2 \to C_1 \to C_0 \to 0$

"good" ($C_i = \bigoplus_{I_i} A_\blacksquare[S]$)

$$R\mathrm{Hom}_A(A_\blacksquare[S], C_*) \xrightarrow{\simeq} R\mathrm{Hom}_A(A[S], C_*)$$

$$\| $$

$$R\mathrm{Hom}_A(A \otimes_R R_\blacksquare[S], C_*)$$

Moreover

$$\begin{tikzcd}
A \otimes_R R_\blacksquare[S] \arrow[r] \arrow[d, equal] & A_\blacksquare[S] \arrow[d, equal] \\
A \otimes_R \prod_I R \arrow[r, hook] & \prod_I A
\end{tikzcd}$$

$\Rightarrow R\mathrm{Hom}\left((\prod_I A) / (A \otimes_R \prod_I R), C_* \right) \simeq 0$

And this will follow from

Lem ... / A ... $R((t^{-1}))$

being a module / $A_\infty := R((T))$

& many observations.

<u>Obs 1</u>: $0 \to \overbrace{R[[U]]}^{=\prod_I R} \otimes_R A \xrightarrow{UT-1} R[[U]] \otimes_R A \to A_\infty \to 0$

s/e/s in $(A,R)_\blacksquare$ -modules

$\Rightarrow$ A_∞ compact in $D((A,R)_\blacksquare)$

<u>Obs 2</u>: $A_\infty \overset{L}{\otimes}_{(A,R)_\blacksquare} A_\infty \xrightarrow[\sim]{mult} A_\infty$

Pf: Computation using <u>Obs 1</u> $\square$

$\Rightarrow$ $D(A_\infty) \hookrightarrow D((A,R)_\blacksquare)$ w/ M in the im $\Leftrightarrow$ $M \xrightarrow{\sim} M \overset{L}{\otimes}_{(A,R)_\blacksquare} A_\infty$

<u>Obs 3</u>: C_* good $\Rightarrow$ $\underline{RHom}_A(A_\infty, C_*) \simeq 0$

<u>Pf</u>: A_∞ compact $\Rightarrow$ $C_* = A[0]$

Use the resentation of Obs 1

$$\begin{array}{ccc} R\underline{\mathrm{Hom}}_R(R[[U]], A) & \xrightarrow{UT-1} & R\underline{\mathrm{Hom}}_R(R[[U]], A) \\ \| & & \| \\ A[U^{-1}] & \xrightarrow{UT-1} & A[U^{-1}] \end{array}$$

UT is ess. nilpotent $\Rightarrow$
$UT-1$ is invertible $\square$

To finish the pf, need

$(\prod_I A)/(A \otimes_R \prod_I R)$ A_∞-module

(Obs 3 $\Rightarrow$ $R\underline{\mathrm{Hom}}_A(A_\infty\text{-mod}, C_*) \simeq 0$)

Coker of $\underset{\substack{\| \\ R[\epsilon]}}{A} \otimes_R \prod_I R \to \prod_I A$

is the same as coker of

$R((T^{-1})) \otimes_{R[[T^{-1}]]} \prod_I R[[T^{-1}]] \to \prod_I R((T^{-1}))$

A_∞

which A_∞-modules

$\square$

$$j : (A,R)_\blacksquare \longrightarrow A_\blacksquare \qquad \left(\prod_I A \in \mathrm{Mod}_{(A,R)_\blacksquare} \right)$$

$\Rightarrow$ get

$$j^* : D((A,R)_\blacksquare) \rightleftarrows D(A_\blacksquare) : j_*$$

$\underline{\text{Thm}}$: j^* has a left adj $j_!$ &

$$\boxed{j_! j^* M := M \overset{L}{\otimes}_{(A,R)_\blacksquare} (A_\infty / A)[-1]}$$

$\underline{\text{Pf}}$:

$$\begin{array}{c}
R\underline{\mathrm{Hom}}_A\left(M \overset{L}{\otimes}_{(A,R)_\blacksquare} (A_\infty/A)[-1],\ N\right) \\
\simeq \big\downarrow (*) \\
R\underline{\mathrm{Hom}}_A\left(M \overset{L}{\otimes}_{(A,R)_\blacksquare} (A_\infty/A)[-1],\ N \overset{L}{\otimes}_{(A,R)_\blacksquare} A_\blacksquare\right) \\
\simeq \big\uparrow (\dagger) \\
R\underline{\mathrm{Hom}}_A\left(M,\ N \overset{L}{\otimes}_{(A,R)_\blacksquare} A_\blacksquare\right) = R\underline{\mathrm{Hom}}_A(j^*M, -''-) \\
\big\uparrow \\
R\underline{\mathrm{Hom}}_A\left(M \overset{L}{\otimes}_{(A,R)_\blacksquare} A_\infty,\ N \overset{L}{\otimes}_{(A,R)} A_\blacksquare\right) \simeq 0
\end{array}$$

DT

($\Rightarrow j_!$ well defined)

$\underline{\text{Obs 3}} \Rightarrow (\dagger)$ equiv

(*) is equiv:

Cone $\left(N \to N \overset{L}{\otimes}_{(A,R)_\blacksquare} A_\blacksquare\right)$ is an A_∞-mod

but $R\underline{Hom}_A\left(M \overset{L}{\otimes}_{(A,R)_\blacksquare} (A_\infty/A)[-1],\ A_\infty\text{-mod}\right) \simeq 0$

$\square$

$f : Spec(A) \to Spec(R)$

Thm 3: a) $f_! : D(A_\blacksquare) \xrightarrow{j_!} D((A,R)_\blacksquare) \xrightarrow{\text{forget}} D(R_\blacksquare)$

preserves colimits, compact objects &

$f_!\left(f^*(M) \overset{L}{\otimes}_{A_\blacksquare} N\right) = M \overset{L}{\otimes}_{R_\blacksquare} f_!(N)$ (P-F)

b) $f_!$ has right adj. $f^!$, $f^!$ comm. w/ $\bigoplus_I$, &

$$f^! M = f^*(M) \overset{L}{\otimes}_{A_\blacksquare} \underbrace{f^!(A)}_{\simeq A[1]}$$

Pf: preserves colims: ✓

compacts nadled

Obs 4: $j_! \prod_I A = \prod_I (A_\infty/A)[-1]$ & compact / $R_\bullet$

Pf: $j_! \prod_I A = j_! j^*(A \otimes_R \prod_I R)$

$= (A \otimes_R \prod_I R) \otimes_{(A,R)_\bullet} (A_\infty/A)[-1]$

$= \prod_I R \overset{L}{\otimes}_{R_\bullet} (A_\infty/A)[-1]$

$= \prod_I (A_\infty/A)[-1]$ $\square$

(P-F): Both sides comm. w/ colims

w.l.o.g $M = \prod_I R$, $N = \prod_J A$

LHS: $f_!(\prod_I A \overset{L}{\otimes}_{A_\bullet} \prod_J A)$

$= f_!(\prod_{J \times I} A) = \prod_{J\times I} (A_\infty/A)[-1]$

RHS: $\prod_I R \overset{L}{\otimes}_{R_\bullet} \prod_J (A_\infty/A)[-1]$

$= \prod_{I\times J} (A_\infty/A)[-1]$

b) $f^! \exists$ ✓

$f^!$ commutes w/ $\oplus$ (adjunction + $f^!$ pres. compacts)

Let's compute $f^!(R)$

$$f^!(R) \simeq \underline{\mathrm{RHom}}_A(A, f^!(R))$$

$$\simeq \underline{\mathrm{RHom}}_R(f_!(A), R)$$

$$\simeq \underline{\mathrm{RHom}}_R(\underbrace{(A_\infty/A)[-1]}_{\prod_{i<0} R\cdot t^i}, R)$$

$$\simeq A[1] \qquad \square$$