VII Analytic rings

Recall: $\mathrm{Cond(Ab)} \underset{\longleftarrow}{\overset{(-)^{\blacksquare}}{\longrightarrow}} \mathrm{Solid}$

$$= \{ C \in \mathrm{Cond(Ab)} \mid \mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[S], C) \}$$

$S \in \mathrm{EDS}$

$\mathbb{Z}[S]^{\blacksquare} = \lim_i \mathbb{Z}[S_i]$ $\quad (S = \lim_i S_i)$ S_i finite

Goal. Develop "relative" theory over condensed rings equipped w/ "analytic structure".

$$\begin{array}{ccccc} \mathrm{Cond(Ab)} = \mathrm{Mod}^{\mathrm{cond}}_{\mathbb{Z}} & & (A, \underline{A}) & & \mathrm{Mod}_{\underline{A}} \\ \cup \qquad\qquad \cup & & \rightsquigarrow & & \downarrow\dashv\cup \\ \mathrm{Solid} = \mathrm{Mod}^{\mathrm{cond}}_{\mathbb{Z}_{\blacksquare}} & & & & \mathrm{Mod}_{\underline{A}} \end{array}$$

Plan:
- Analytic rings in [Condensed Mathematics]
  - abelian category & its derived category
  - some examples, e.g. for $\mathbb{Z}_p$, A discrete
  - not suitable for general Huber rings, e.g. $(\mathbb{Q}_p, \mathbb{Z}_p)$
- A glimpse into analytic rings in [Analytic Geometry]
  - better general theory, but $D(\mathcal{A}, \underline{\mathcal{A}}) \neq D(\mathcal{D}(\mathcal{A}, \underline{\mathcal{A}})^{\heartsuit})$

Def. A pre-analytic ring is a triple $(\mathcal{A}, \mathcal{A}[-], \eta)$ where
- $\underline{\mathcal{A}} \in \mathrm{Cond(Ring)}$
- $\mathcal{A}[-] : \mathrm{EDS} \to \mathrm{Mod}_{\underline{\mathcal{A}}}$, $S \mapsto \mathcal{A}[S]$ ($\underline{\mathcal{A}}$-modules in Cond(Ab))

  s.t. $\mathcal{A}[\bigsqcup_{\text{finite}} S_i] \xleftarrow{\simeq} \bigoplus \mathcal{A}[S_i]$
- $\eta : S \to \mathcal{A}[S]$ nat. trafo

Ex. • $\mathbb{Z}_{p\blacksquare} = (\underset{\mathbb{Z}_{p\blacksquare}}{\underline{\mathbb{Z}}_p}, \mathbb{Z}_{p\blacksquare}[-])$

$\mathbb{Z}_{p\blacksquare}[S] = \mathbb{Z}_p[S]^{\blacksquare} = \lim_i \mathbb{Z}_p[S_i]$ $\quad$ w/ $S = \lim_i S_i$ finite

- A discrete ring $\rightsquigarrow (A, \mathbb{Z})_\blacksquare$ w/ $S \mapsto \mathbb{Z}_\blacksquare[S] \otimes_{\mathbb{Z}} A$

- $A/\mathbb{Z}$ fin. gen. $\rightsquigarrow A_\blacksquare$ w/ $S \mapsto \lim_i A[S_i]$.

Def. A pre-analytic ring $\mathcal{A}$ is called **analytic** iff for all $C \in Ch_{\geq 0}(\mathrm{Mod}_{\mathcal{A}})$ w/ $C_i = \bigoplus_{j \in I_i} \mathcal{A}[T_j]$ have

$$\boxed{R\underline{\mathrm{Hom}}(\mathcal{A}[S], C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathcal{A}[S], C)}$$

Recall (categorical lemma, Marc's talk)

Let
- $\mathcal{A}$ = abelian category w/ all colimits
- $\mathcal{A}_0 \subset \mathcal{A}$ = subcat. of compact projective objects generating $\mathcal{A}$
- $F : \mathcal{A}_0 \to \mathcal{A}$ functor w/ nat. trafo $X \to F(X)$

s.t.
- $\forall\, C \in Ch_{\geq 0}(\mathcal{A}_0)$ w/ $C_i = \bigoplus_{j \in I_i} F(Y_j)$ $\forall X \in \mathcal{A}_0 : R\mathrm{Hom}(F(X), C) \xrightarrow{\sim} R\mathrm{Hom}(X, C)$.

Then the subcat $\mathcal{A}_F = \{ Y \in \mathcal{A} \mid \forall X \in \mathcal{A}_0 : \mathrm{Hom}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}(X, Y) \}$
- is stable under limits, colimits, and extensions,
- $\{ F(X) \mid X \in \mathcal{A}_0 \}$ are compact projective generators of $\mathcal{A}_F$, and
- have adjunction $L : \mathcal{A} \rightleftarrows \mathcal{A}_F$ extending $F : \mathcal{A}_0 \to \mathcal{A}_F$.

and for $D_F(\mathcal{A}) = \{ C \in D(\mathcal{A}) \mid \forall X \in \mathcal{A}_0 : R\mathrm{Hom}(F(X), C) \xrightarrow{\sim} R\mathrm{Hom}(X, C) \}$

have
- $D(\mathcal{A}_F) \xhookrightarrow{\text{fully faithful}} D(\mathcal{A}) \supset D_F(\mathcal{A})$, $\quad D(\mathcal{A}_F) \xrightarrow{\sim} D_F(\mathcal{A})$
- $C \in D_F(\mathcal{A}) \iff \forall i : H^i(C) \in \mathcal{A}_F$
- have adjunction $L : D(\mathcal{A}) \rightleftarrows D_F(\mathcal{A})$.

Def. Let $\mathcal{A}$ be an analytic ring.

- $\mathrm{Mod}_{\underline{\mathcal{A}}} \supset \mathrm{Mod}_{\mathcal{A}} := \{ M \mid \forall S \in \mathrm{EDS}: \mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M) \xrightarrow{\sim} \underbrace{\mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], M)}_{=M(S)} \}$
- $D(\mathcal{A}) := D(\mathrm{Mod}_{\mathcal{A}})$

Prop. Let $\mathcal{A}$ be an analytic ring. Then:

- $\mathrm{Mod}_{\mathcal{A}}$ is abelian and stable under limits, colimits, and extensions in $\mathrm{Mod}_{\underline{\mathcal{A}}}$
- $\{\mathcal{A}[S] \mid S \in \mathrm{EDS}\}$ is fam. of compact projective generators of $\mathrm{Mod}_{\mathcal{A}}$
- $\exists$ adjunction $\mathrm{Mod}_{\underline{\mathcal{A}}} \underset{}{\overset{L}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{A}}$, $M \mapsto M \otimes_{\underline{\mathcal{A}}} \mathcal{A}$.

If $\underline{\mathcal{A}}$ is commutative, there exists unique $\otimes_{\mathcal{A}}$ on $\mathrm{Mod}_{\mathcal{A}}$
s.t. $\mathrm{Mod}_{\underline{\mathcal{A}}} \xrightarrow{L} \mathrm{Mod}_{\mathcal{A}}$ is sym. monoidal.

The functor $D(\mathrm{Mod}_{\mathcal{A}}) \longrightarrow D(\mathrm{Mod}_{\underline{\mathcal{A}}})$ is fully faithful
w/ essential image $\{C \in D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \mid \forall S \in \mathrm{EDS}: R\mathrm{Hom}(\mathcal{A}[S], C) \xrightarrow{\sim} R\mathrm{Hom}(\underline{\mathcal{A}}[S], C)\}$

- $C \in D(\mathrm{Mod}_{\mathcal{A}}) \iff \forall i: H^i(C) \in \mathrm{Mod}_{\mathcal{A}}$
- have adjunction $D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \overset{L}{\rightleftarrows} D(\mathrm{Mod}_{\mathcal{A}})$, $C \mapsto C \otimes^L_{\underline{\mathcal{A}}} \mathcal{A}$

If $\underline{\mathcal{A}}$ is commutative, there exist unique $\otimes^L_{\mathcal{A}}$ on $D(\mathrm{Mod}_{\mathcal{A}})$
s.t. $D(\mathrm{Mod}_{\underline{\mathcal{A}}}) \xrightarrow{L} D(\mathrm{Mod}_{\mathcal{A}})$ is symmetric monoidal.

Proof. essentially the categorical lemma from above. $\square$

Ex. The pre-analytic ring $\mathbb{Z}_\blacksquare$ is analytic and have

$$\begin{tikzcd}
M \arrow[d] & \in & \mathrm{Mod}_{\mathbb{Z}} \arrow[d, phantom, "\cup"] & = & \mathrm{Cond}(\mathrm{Ab}) \arrow[d, phantom, "\cup"] & \ni & M \arrow[d] \\
M \otimes_{\mathbb{Z}} \mathbb{Z}_\blacksquare & & \mathrm{Mod}_{\mathbb{Z}_\blacksquare} & = & \mathrm{Solid} & & M^\blacksquare
\end{tikzcd}$$

as well as $C^{L\blacksquare} = C \otimes^L_{\mathbb{Z}} \mathbb{Z}_\blacksquare$

Def. Let $\mathcal{A},\mathcal{B}$ be analytic rings. A map of analytic rings $f:\mathcal{A}\to\mathcal{B}$ is a map $f:\underline{A}\to\underline{B}$ in $\mathrm{Cond}(\mathrm{Ring})$ st $\forall S\in \mathrm{EDS}$: $\mathcal{B}[S]\in \mathrm{Mod}_{\mathcal{A}}(\mathrm{CMod}_{\underline{A}})$.

Note: If $f:\mathcal{A}\to\mathcal{B}$ is a map of analytic rings, then $\forall S\in \mathrm{EDS}$:

$$\begin{tikzcd} & \mathcal{A}[S] \arrow[d, dashed, "\exists!"] \\ S \arrow[ur] \arrow[r] & \mathcal{B}[S] \end{tikzcd}$$

Get nat. trafo $\mathcal{A}[-]\to\mathcal{B}[-]$.

Prop. Let $f:\mathcal{A}\to\mathcal{B}$ be a map of analytic rings. Then

- $$\begin{tikzcd} \mathrm{Mod}_{\underline{A}} \arrow[r, "-\otimes_{\underline{A}}\underline{B}"] \arrow[d, "-\otimes_{\underline{A}}\mathcal{A}"'] & \mathrm{Mod}_{\underline{B}} \arrow[d, "-\otimes_{\underline{B}}\mathcal{B}"] \\ \mathrm{Mod}_{\mathcal{A}} \arrow[r, "\exists!\ -\otimes_{\mathcal{A}}\mathcal{B}"] & \mathrm{Mod}_{\mathcal{B}} \end{tikzcd}$$

  $\boxed{\begin{array}{l} M\in \mathrm{Mod}_{\underline{A}} \rightsquigarrow M\otimes_{\underline{A}}\underline{B}\in\mathrm{Mod}_{\underline{B}} \\ \rightsquigarrow (M\otimes_{\underline{A}}\underline{B})\otimes_{\underline{B}}\mathcal{B}\in\mathrm{Mod}_{\mathcal{B}}. \\ \rightsquigarrow N\in\mathrm{Mod}_{\mathcal{A}} \rightsquigarrow (N\otimes_{\underline{A}}\underline{B})\otimes_{\underline{B}}\mathcal{B} \end{array}}$

- $$\begin{tikzcd} D(\mathrm{Mod}_{\underline{A}}) \arrow[r, "-\otimes^L_{\underline{A}}\underline{B}"] \arrow[d, "-\otimes^L_{\underline{A}}\mathcal{A}"'] & D(\mathrm{Mod}_{\underline{B}}) \arrow[d, "-\otimes^L_{\underline{B}}\mathcal{B}"] \\ D(\mathrm{Mod}_{\mathcal{A}}) \arrow[r, "\exists!\ -\otimes^L_{\mathcal{A}}\mathcal{B}"] & D(\mathrm{Mod}_{\mathcal{B}}) \end{tikzcd}$$

Proof • as the $\mathcal{B}[S]$ generate $\mathrm{Mod}_{\mathcal{B}}$

• as $C\in D(\mathrm{Mod}_{\mathcal{A}}) \Leftrightarrow \forall i: H^i(C)\in\mathrm{Mod}_{\mathcal{A}}$.

Prop. Let $\mathcal{A},\mathcal{B}$ be analytic rings and $f:\underline{A}\to\underline{B}$ map of underlying condensed rings.

Assume for all $\mathrm{EDS}\ni S\to\underline{A}\to\underline{B}$ that

$\mathcal{A}[-]\to\mathcal{B}[-]$ in $\mathrm{Mod}_{\mathcal{A}}$

$$\begin{tikzcd} \mathcal{A}[S] \arrow[r, Rightarrow] \arrow[d, "\exists!"'] \arrow[dr, phantom, "="] & \mathcal{A}[*] \text{ in } \mathrm{Mod}_{\mathcal{A}} \arrow[d, "\exists!"] \\ \mathcal{B}[S] \arrow[r] & \mathcal{B}[*] \text{ in } \mathrm{Mod}_{\mathcal{B}} \end{tikzcd}$$

Old def $\mathcal{A}[-]\to\mathcal{B}[-]$ in $\ldots(\mathrm{Mod}_{\mathcal{B}})$ st

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \mathcal{A}[S] \arrow[d] \arrow[dl, phantom, "="] \\ & \mathcal{B}[S] \end{tikzcd}$$

Then f is a map of analytic rings $\mathcal{A}\to\mathcal{B}$.

Prop. The pre-anal. rings $\underbrace{\mathbb{Z}_{p,■} \text{ and } (A,\mathbb{Z})_■}_{=\mathcal{A}}$ (for A discrete) are analytic.

$\left(\text{w/ } \underline{\mathcal{A}} \leftarrow \begin{matrix} \mathbb{Z}_p \\ A \end{matrix}\right)$

Proof. Know $\mathbb{Z}_■$ is an analytic ring.

- WTS: for $C \in Ch_{\geq 0}(Mod_{\underline{\mathcal{A}}})$ be a "test complex", $S \in EDS$

$$R\underline{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \overset{\sim}{\underset{?}{\longrightarrow}} R\underline{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C) = R\underline{Hom}(\mathbb{Z}[S], C)$$

$$\boxed{\begin{array}{c} \underline{\mathcal{A}}[S] \Rightarrow \mathcal{A}[S] \\ \uparrow \nearrow \\ S \end{array}}$$

- Have $\mathcal{A}[S] \overset{(*)}{=} \mathbb{Z}_■[S] \otimes^L_{\mathbb{Z}} \underline{\mathcal{A}}$ $\left(\begin{array}{l} \mathbb{Z}_{p,■}[S] = \lim \mathbb{Z}_p[S_i] \\ (A,\mathbb{Z})_■[S] = \mathbb{Z}_■[S] \otimes_{\mathbb{Z}} A \end{array} \right)$

- $$\begin{array}{ccc} R\underline{Hom}(\mathbb{Z}_■[S], C) & = & R\underline{Hom}_{\underline{\mathcal{A}}}(\mathbb{Z}_■[S] \otimes^L_{\mathbb{Z}} \underline{\mathcal{A}}, C) \\ \| \text{ - } C \text{ solid} & & \| (*) \\ R\underline{Hom}(\mathbb{Z}[S], C) & = & R\underline{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \end{array}$$

☐

$$\mathbb{Z}_{p,■}[S] = \lim_i \mathbb{Z}_p[S_i] = \lim_i (\lim_n \mathbb{Z}/p^n)[S_i]$$
$$= \lim_n \lim_i \mathbb{Z}/p \quad ?$$
$\quad$ ↳ perfect

What about $\mathbb{R}$? For $\mathbb{R} \in Cond(Ring)$

⇝ pre-anal. ring $\mathbb{R}_{\ell^p}$ for $\boxed{0 < p \leq 1}$

w/ $$\mathbb{R}_{\ell^p}[S] = \bigcup_{r>0} \lim_i \underbrace{\mathbb{R}[S_i]_{\ell^p \leq r}}_{\{(x_j)_j \in \mathbb{R}[S_i] \,\mid\, (\sum_j |x_j|^p)^{1/p} \leq r\}}$$

Prop. $\mathbb{R}_{\ell^p}$ is not an analytic ring.

Proof. For $\underline{A}$ anal. ring have for all $S, T \in \mathrm{EDS}$, $i > 0$

$$\mathrm{Ext}^i_{\underline{A}}(\underline{A}[S], \underline{A}[T]) = H^i(S, \underline{A}[T]) = 0$$

$\llcorner$ S extremally disconnected.

But for $S \in \mathrm{EDS}$ have $\mathrm{Ext}^1_{\mathbb{R}}(\mathbb{R}_{\ell^p}[S], \mathbb{R}_{\ell^p}[S]) \neq 0$

since there ex. ext. $0 \to \mathbb{R}_{\ell^p}[S] \to \mathcal{E} \to \mathbb{R}_{\ell^p}[S] \to 0$.

Thm [AG, Thm 6.5] For $p \leq 1$

$$\mathbb{R}_{\ell^{<p}} := \operatorname*{colim}_{q<p} \mathbb{R}_{\ell^q} \text{ is analytic.}$$

$\square$

What about Huber rings? (A, A^+) — top. ring; integrally closed top. subring.)

$\simeq (A_0[\tfrac{1}{\pi}], A_0)$ — π-adic, π top. nilpot. ring.

e.g. $(\mathbb{Q}_p, \mathbb{Z}_p)$, $(\mathbb{F}_p((t)), \mathbb{F}_p[[t]])$, $A = \dfrac{k\langle t_1, \ldots, t_n\rangle}{(f_1, \ldots, f_k)}$

Prop $\exists$ $\mathrm{HubRings} \hookrightarrow \mathrm{AnRing}$

w/

- $R/\mathbb{Z}$ fin. gen. $\rightsquigarrow$ $R_{\blacksquare}[S] := \lim_i R[S_i]$
- $R/\mathbb{Z}$ discrete $\rightsquigarrow$ $R_{\blacksquare} = \operatorname*{colim}_{\substack{R' \to R \\ \text{fin. gen}}}$
- $(A, A^+)_{\blacksquare} = \underline{A} \otimes (A^+_{\mathrm{disc}})_{\blacksquare}$

  A^+_{discrete}

$\boxed{\mathrm{AnRing}} = \infty$-cat. of $\underbrace{\text{normalised}}$ $\overset{\text{analytic}}{\text{animated}}$ comm. rings

$\underline{A} \rightsquigarrow \underline{A}[*]$ $\qquad$ $\underline{A} \to \underline{A}' =$ normalised

$$\mathrm{Ani}(\mathrm{Ab}) \simeq \mathcal{D}_{\geq 0}(\mathrm{Ab}) \\ \cup \\ \mathcal{D}_{\geq}(\underline{A})$$

Want $\underline{A}[*]$ comm if $\underline{A}$ comm ring.