

## XI. Coherent duality

Grothendieck duality: Let  $f: X \rightarrow \text{Spec } R$  be a **smooth & proper** morph. of schemes of relative dimension  $d$ . Set  $\omega_{X/R} := \wedge^d \Omega_{X/R}^1$ .

Then there exists a **trace map**

$$tr: H^d(X, \omega_{X/R}) \rightarrow R$$

s.t. for all  $F \in D_{\text{qcoh}}(X)$  have

$$R\text{Hom}_{\mathcal{O}_X}(F, \omega_{X/R})[d] \xrightarrow{\sim} R\text{Hom}_R(R\Gamma(X, F), R) \text{ in } D(R).$$

Goal: Generalise this to the **non-proper** case

Def. For scheme  $X$  set  $D(\mathcal{O}_{X, \square}) := D(\mathcal{O}_{X^{\text{ad}}, \mathcal{O}_{X^{\text{ad}}}^+})$

Main Thm. Let  $f: X \rightarrow \text{Spec } R$  be a separated, **smooth** morph. of finite type of relative dimension  $d$ .

Then:

1) There exists a canonical functor

$$f_! : D(\mathcal{O}_{X, \square}) \rightarrow D(R)$$

2) If  $f$  is proper, then  $f_! = R\Gamma(X, -)$

3) The functor  $f_!$  preserves compact objects

4) There exists a trace map  $tr: f_! \omega_{X/R}[d] \rightarrow R$

$$\text{w/ } \omega_{X/R} = \wedge^d \Omega_{X/R}^1$$

s.t. for all  $C \in \mathcal{D}(O_{X, \mathbb{A}^1})$  the map

$$R\mathrm{Hom}_{O_{X, \mathbb{A}^1}}(C, \omega_{X/R})[d] \xrightarrow{\sim} R\mathrm{Hom}_R(f_! C, R)$$

is an isomorphism.

In order to prove this, we establish a

$$\underbrace{\otimes_X \dashv \mathrm{Hom}}_{\text{pink underline}}, \quad \underbrace{f^* \dashv f_*}_{\text{pink underline}}, \quad \boxed{f_! \dashv f^!}$$

Most of the work for the construction of  $f_!$  was already done in Toni's talk:

Recall. For map  $R \rightarrow A$  of finitely generated  $\mathbb{Z}$ -algebras the pre-analytic ring

$$(A, R)_{\mathbb{A}^1} \text{ is given by } \mathrm{EDS} \ni S \mapsto (A, R)_{\mathbb{A}^1}[S] := A \otimes_R^L \varinjlim_{S_i \text{ finite}} R[S_i]$$

Thm [CM, Thm 8.13] Let  $A \rightarrow R$  map of fin. gen.  $\mathbb{Z}$ -algs.

i)  $(A, R)_{\mathbb{A}^1}$  is an analytic ring

ii) For  $R \rightarrow S \rightarrow A$  have adjunctions

$$\begin{array}{ccc} \mathcal{D}((A, S)_{\mathbb{A}^1}) & \begin{array}{c} \xleftarrow{j_!} \\ \xleftarrow{j^*} \perp \\ \xrightarrow{j_*} \perp \\ \xrightarrow{J_* = \text{forget}} \end{array} & \mathcal{D}((A, R)_{\mathbb{A}^1}) \end{array}$$

For  $f: \mathrm{Spec} A \rightarrow \mathrm{Spec} R$  set

$$f_! : \mathcal{D}(A_{\mathbb{A}^1}) = \mathcal{D}(A, A)_{\mathbb{A}^1} \xrightarrow{j_!} \mathcal{D}(A, R)_{\mathbb{A}^1} \xrightarrow{\text{forget}} \mathcal{D}(R_{\mathbb{A}^1}).$$

iii) The functor  $f_!$  preserves colimits & compact objects.

iv) Have adjunction

$$f_! : \mathcal{D}(A_{\mathbb{A}^1}) \begin{array}{c} \xrightarrow{\perp} \\ \xleftarrow{\perp} \end{array} \mathcal{D}(R_{\mathbb{A}^1}) : f^!$$

Have explicit formulae:

$$\bullet j_! j^* M = M \otimes_{(A,R)}^L \underbrace{j_! A}_{\cong \left( \frac{R((t^{-1}))}{R[t]} \right) [1]}$$

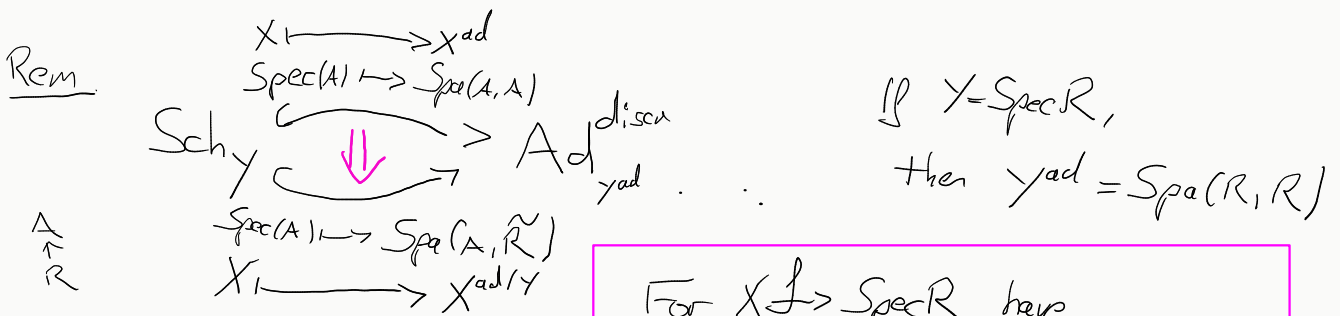
$$\bullet j_! (j^* M \otimes_{A}^L N) = M \otimes_{R}^L j_! N$$

$$\bullet j_! M = j^* M \otimes_{A}^L \underbrace{j_! R}_{\cong A[1]}$$

As a consequence we get:

Lemma 1. For sep. morph of schemes  $f: X \rightarrow Y$  of finite type have an adjunction

$$j_! : D(\mathcal{O}_{X^{\text{ad}}}, \mathcal{O}_{X^{\text{ad}}}) \xleftarrow{j^*} D(\mathcal{O}_{X^{\text{ad}}/Y}, \mathcal{O}_{X^{\text{ad}}/Y}^+) : j^*$$



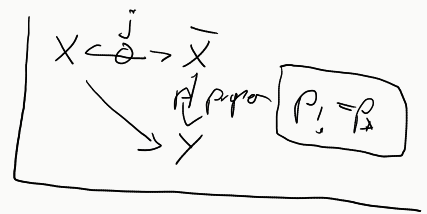
For  $X \rightarrow \text{Spec } R$  here  
 $j_! = X^{\text{ad}} \hookrightarrow X^{\text{ad}}/R$   
 If  $f$  proper then  $j_!$  is an isom.

underlying sets:

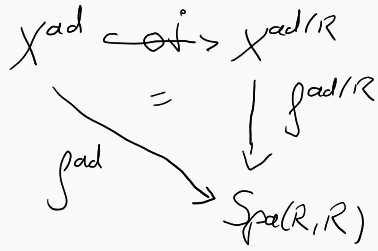
$$|X^{\text{ad}}| \cong \left\{ \begin{array}{l} \text{Spec } V \longrightarrow X \\ \text{val ring} \end{array} \right\} / \cong$$

$$|X^{\text{ad}}/R| \cong \left\{ \begin{array}{l} \text{Spec } \text{Frac}(V) \longrightarrow X \\ \downarrow \\ \text{Spec } V \longrightarrow \text{Spec } R \\ \text{val ring} \end{array} \right\} / \cong$$

Proof of Main Thm. [11-3]  $X \xrightarrow{f} \text{Spec } R$



1) Have factorisation



Define

$$f_! := \int_{\star}^{ad/R} \circ \int_!$$

2) If  $f$  proper  $\Rightarrow f$  isom.  $\Rightarrow f_! = \int_{\star}^{ad/R} = \text{RM}(X, -)$

3) HIS:  $f_!(\text{compact}) = \text{compact} \rightarrow$  can check this locally for which it was proved in Torii's talk

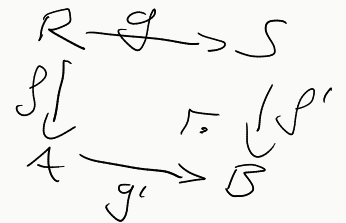
□

Lemma 2. Let  $f: R \rightarrow A$  be a map of rings that is the base change of a fin. gen. map of finite tor dimension b/w noeth. rings.

i) The nat. map  $f_! R \otimes_{A_{\#}}^L f^{\star} \rightarrow f_! : D(R_{\#}) \rightarrow D(A_{\#})$  is an equiv.

ii) For flat map  $g: R \rightarrow S$  have an equiv.

$$g_!^{\star} f_! \xrightarrow{\sim} f_! \circ g^{\star} : D(R_{\#}) \rightarrow D(A_{\#})$$



$$f_! \rightarrow f_! g^{\star} g^{\star} \xrightarrow{\sim} g_!^{\star} f_! g^{\star}$$

$\hookrightarrow$  adjoint to proper base change isom ( $f_! g^{\star} \xrightarrow{\sim} g_!^{\star} f_!$ )

using (PF)

Proof. Two cases:  $f$  surjective OR  $A = R[t]$

$$\boxed{f \text{ surjective}} \xrightarrow{\text{proper}} f_! = f_{\star} \subseteq \text{RHom}_R(A, R) \otimes_{A_{\#}}^L f_!(-)$$

$$\perp \qquad \perp$$

$$f_! = \text{RHom}_R(A, -) \leftarrow \varphi \text{RHom}_R(A, R) \otimes_{R_{\#}}^L (-)$$

Claim:  $\varphi$  equiv.

$\hookrightarrow$  By assumption,  $A$  is perfect  $\rightarrow A \simeq (C_1 \xrightarrow{\text{fin. gen. proj}} \dots \rightarrow C_n) \rightarrow$  can reduce to the case  $A = R$

Observe that everything is compatible w/ flat base change.

$[A=R/I]$  omitted.

□

Lemma. Let  $f: R \rightarrow A$  be a regular closed immersion (ie.  $f$  seq. & locally  $I = \ker(f)$  is gen. by a regular seq.  $f_1, \dots, f_c$ ) which is everywhere of pure codim  $c$ .

Then there ex. a nat. isom.

$$f^! R = R \text{Hom}_R(A, R) \cong (\wedge^c \mathbb{F}_{I^2})^\vee [-c]$$

Proof. Locally,  $A = R / (f_1, \dots, f_c)$  w/  $f_1, \dots, f_c$  regular sequence.

Get

$$R \text{Hom}_R(A, R) \cong A[-c] \xrightarrow{\cong} (\wedge^c \mathbb{F}_{I^2})^\vee [-c]$$

$\uparrow$   
 Koszul  
 computation  
 $\mapsto (f_1, \dots, f_c)^\vee$

Can check that this compos. does not depend on choice of  $f_1, \dots, f_c$ .  
 $\leadsto$  can glue the local isos.

□

Thm A. Let  $f: A \rightarrow R$  be smooth of rel. dim  $d$ .

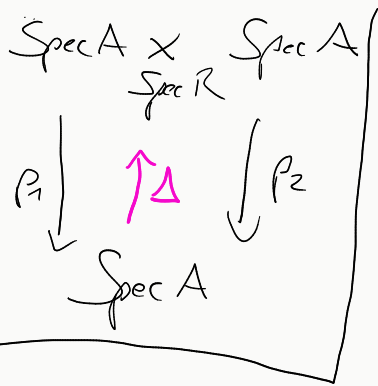
Then  $f^! R \cong (\wedge^d \Omega_{A/R}^1)[d]$

Proof. • have reg. closed imm.  $\text{Spec}(A) \hookrightarrow \mathbb{A}_R^n$ .

• Claim:  $f^! R$  is line bundle concentrated in degree  $d$ .

↳ affine space:  $f^! R = R[t_1, \dots, t_n][n]$  ( $d = n - c$ )

↳ reg. closed imm  $f^! R \cong (\wedge^c \mathbb{F}_{I^2})^\vee [-c]$  (Len 3)



Can compute:

$$\begin{aligned}
 f^!R &= \Delta^! p_1^! f^!R \quad (\text{p}_1^! \text{ is twist of } p_2^!) \\
 &= \Delta^! \left( p_1^* f^!R \otimes_{(A \otimes_R A)}^L p_1^! A \right) \\
 &= p_1^! f^!R
 \end{aligned}$$

$$\stackrel{(\text{Lem 2.ii})}{=} \Delta^! \left( p_1^* f^!R \otimes_{(A \otimes_R A)}^L p_2^* f^!R \right)$$

$$= \Delta^! (\dots) \otimes_{A \otimes_R A}^L \Delta^! (A \otimes_R A)$$

$$= f^!R \otimes_{A \otimes_R A}^L f^!R \otimes_{A \otimes_R A}^L \Delta^! (A \otimes_R A)$$

$$\begin{aligned}
 \Rightarrow f^!R &\cong (\Delta^! (A \otimes_R A))^{\vee} \cong (\wedge^d \mathbb{I}_{\mathbb{I}^2})^{\vee} [-d] \\
 &\cong \Omega_{A/R}^1 \cong \mathbb{I}_{\mathbb{I}^2}
 \end{aligned}$$

$(\Delta^! \text{ is twist of } \Delta^*)$

□

Cor: For  $X \rightarrow \text{Spec } R$  as in Main Thm we have

$$f^!R \cong \underbrace{(\wedge^d \Omega_{X/R}^1)}_{= \omega_{X/R}} [d]$$

Proof of Main Thm [4] WTS: 
$$\frac{f_! \omega_{X/R} [d] \xrightarrow{tr} R}{\omega_{X/R} [d] \xrightarrow{\sim} f^!R} \text{ adj.}$$

For  $C \in D(\mathcal{O}_{X, \mathbb{I}^2})$  consider

$$\begin{aligned}
 R\text{Hom}_{\mathcal{O}_X} (C, \omega_{X/R}) [d] &\longrightarrow R\text{Hom}_R (f_! C, R) \\
 &\xrightarrow{\sim} R\text{Hom}_{\mathcal{O}_X} (C, f^!R)
 \end{aligned}$$

□