Condensed/Pyknotic
Seminar, SS 2020

30/07/20

XI. Coherent duality

Grothendieck duality: Let $f: X \to \operatorname{Spec} R$ be a smooth & proper morph. of schemes of relative dimension d. Set $\omega_{X/R} := \Lambda^d \Omega^1_{X/R}$.

Then there exists a trace map

$$\mathrm{tr}: H^d(X, \omega_{X/R}) \to R$$

s.t. for all $F \in D_{qcoh}(X)$ have

$$R\mathrm{Hom}_{\mathcal{O}_X}(F, \omega_{X/R})[d] \xrightarrow{\sim} R\mathrm{Hom}_R(R\Gamma(X,F), R) \quad \text{in } D(R).$$

Goal: Generalise this to the non-proper case

Def. For scheme X set $D(\mathcal{O}_{X,\blacksquare}) := D((\mathcal{O}_{X^{ad}}, \mathcal{O}^+_{X^{ad}})_\blacksquare)$

Main Thm. Let $f: X \to \operatorname{Spec} R$ be a separated, smooth morph. of finite type of relative dimension d.

Then:

1) There exists a canonical functor

$$f_!: D(\mathcal{O}_{X,\blacksquare}) \to D(R_\blacksquare)$$

2) If f is proper, then $f_! = R\Gamma(X,-)$

3) The functor $f_!$ preserves compact objects

4) There exists a trace map $\mathrm{tr}: f_! \omega_{X/R}[d] \to R$

w/ $\omega_{X/R} = \Lambda^d \Omega^1_{X/R}$

s.t. for all $C \in D(\mathcal{O}_{X,\blacksquare})$ the map

$$R\mathrm{Hom}_{\mathcal{O}_{X,\blacksquare}}(C, \omega_{X/R})[d] \xrightarrow{\sim} R\mathrm{Hom}_R(f_! C, R)$$

is an isomorphism.

In order to prove this, we establish a

$$\underbrace{\otimes_X \dashv \underline{\mathrm{Hom}} \;,\; f^* \dashv f_*}_{\checkmark} \;,\; \boxed{f_! \dashv f^!}$$

Most of the work for the construction of $f_!$ was already done in Toni's talk:

<u>Recall.</u> For map $R \to A$ of finitely generated $\mathbb{Z}$-algebras the pre-analytic ring $(A,R)_\blacksquare$ is given by $\mathrm{EDS} \ni \underset{\substack{\shortparallel \\ \lim_i S_i}}{S} \mapsto (A,R)_\blacksquare[S] := A \otimes^L_R \lim_i R[S_i]$, S_i finite

<u>Thm</u> [CM, Thm 8.13] Let $A \to R$ map of fin. gen. $\mathbb{Z}$-algs.

i) $(A,R)_\blacksquare$ is an analytic ring

ii) For $R \to S \to A$ have adjunctions

$$\begin{tikzcd} D((A,S)_\blacksquare) \arrow[r, bend left, "j_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, bend right, "j_* = \text{forget}"'] & D((A,R)_\blacksquare) \arrow[l, "j^*"'] \end{tikzcd}$$

For $f: \mathrm{Spec}\, A \to \mathrm{Spec}\, R$ set

$$f_! : D(A_\blacksquare) = D((A,A)_\blacksquare) \xrightarrow{j_!} D((A,R)_\blacksquare) \xrightarrow{\text{forget}} D(R_\blacksquare).$$

iii) The functor $f_!$ preserves colimits & compact objects.

iv) Have adjunction

$$f_! : D(A_\blacksquare) \underset{\perp}{\rightleftarrows} D(R_\blacksquare) : f^!$$

Have explicit formulae:

- $$j_! j^* M = M \otimes^L_{(A,R)_\blacksquare} \underbrace{j_! A}_{\underset{(A=R[t])}{=} \left(R((t^{-1})) / R[t] \right)[1]}$$
- $$f_! (f^* M \otimes^L_{A_\blacksquare} N) = M \otimes^L_{R_\blacksquare} f_! N$$
- $$f^! M = f^* M \otimes^L_{A_\blacksquare} \underbrace{f^! R}_{\simeq A[1]}$$

As a consequence we get:

<u>Lemma 1.</u> For sep. morph of schemes $f: X \to Y$ of finite type have an adjunction

$$j_! : D((\mathcal{O}_{X^{ad}}, \mathcal{O}^+_{X^{ad}})_\blacksquare) \underset{j^*}{\overset{\perp}{\rightleftarrows}} D((\mathcal{O}_{X^{ad/Y}}, \mathcal{O}^+_{X^{ad/Y}})_\blacksquare) : j^*$$

<u>Rem.</u>

$$\begin{tikzcd} \mathrm{Sch}_Y \arrow[rr, bend left, hook, "{X \mapsto X^{ad},\ \mathrm{Spec}(A) \mapsto \mathrm{Spa}(A,A)}"] \arrow[rr, bend right, hook, "{\mathrm{Spec}(A) \mapsto \mathrm{Spa}(A,\tilde{R}),\ X \mapsto X^{ad/Y}}"'] \arrow[rr, phantom, "\Downarrow"] & & \mathrm{Ad}^{discr}_{Y^{ad}} \end{tikzcd}$$

$$\begin{tikzcd} A \\ R \arrow[u] \end{tikzcd}$$

If $Y = \mathrm{Spec} R$, then $Y^{ad} = \mathrm{Spa}(R,R)$

For $X \xrightarrow{f} \mathrm{Spec} R$ have
$j: X^{ad} \xhookrightarrow{\circ} X^{ad/R}$
If f proper then j is an isom.

<u>underlying sets:</u>

$$|X^{ad}| \simeq \{ \underset{\text{val ring}}{\mathrm{Spec}\, V} \to X \} / \simeq$$

$$|X^{ad/R}| \simeq \left\{ \begin{tikzcd} \mathrm{Spec}\, \mathrm{Frac}(V) \arrow[r] \arrow[d] & X \arrow[d] \\ \underset{\text{val ring}}{\mathrm{Spec}\, V} \arrow[r] & \mathrm{Spec} R \end{tikzcd} \right\} / \simeq$$

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr] & \overline{X} \arrow[d, "p\ \text{proper}"] \\ & Y \end{tikzcd}$$ $\boxed{p_! = p_*}$

Proof of Main Thm. [11-3]

$X \xrightarrow{f} \operatorname{Spec} R$

1) Have factorisation

$$\begin{tikzcd} X^{ad} \arrow[r, hook, "j"] \arrow[dr, "f^{ad}"'] \arrow[dr, phantom, shift left=3, "="] & X^{ad/R} \arrow[d, "f^{ad/R}"] \\ & \operatorname{Spa}(R,R) \end{tikzcd}$$

Define

$$f_! := f^{ad/R}_* \circ j_!$$

2) If f proper $\Rightarrow$ j isom. $\Rightarrow$ $f_! = f^{ad/R}_* = R\Gamma(X,-)$

3) HTS: $f_!(\text{compact}) = \text{compact}$ ⇝ can check this locally for which it was proved in Toni's talk

$\square$

Lemma 2. Let $f: R \to A$ be a map of rings that is the base change of a fin. gen map of finite Tor dimension b/w noeth. rings.

i) The nat. map $f^! R \otimes^L_{A_\blacksquare} f^* \to f^! : D(R_\blacksquare) \to D(A_\blacksquare)$ is an equiv.

ii) For flat map $g: R \to S$ have an equiv.

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "f'"] \\ A \arrow[r, "g'"'] & B \end{tikzcd}$$

$$\underline{g'^* f^! \xrightarrow{\sim} f'^! g^* : D(R_\blacksquare) \to D(A_\blacksquare)}\ \text{adj}$$

$$f^! \to f^! g_* g^* \simeq g'_* f'^! g^*$$

↳ adjoint to proper base change isom $(f_! g^* \simeq g'^* f'_!)$ using (PF)

Proof. Two cases: f surjective OR $A = R[t]$

$$\begin{tikzcd} \boxed{f\ \text{surjective}} \arrow[r, Rightarrow, "\text{proper}"] & f_! \arrow[r, phantom, "="] \arrow[d, phantom, "\perp"] & f_* \arrow[d, phantom, "\perp"] & & \\ & f^! \arrow[r, phantom, "="] & R\operatorname{Hom}_R(A,-) & \underbrace{R\operatorname{Hom}_R(A,R) \otimes^L_{R_\blacksquare} (-)}_{\simeq R\operatorname{Hom}_R(A,R) \otimes^L_{A_\blacksquare} f^*(-)} \arrow[l, "\varphi"'] \end{tikzcd}$$

Claim: φ equiv.

↳ By assumption, A is perfect ⇝ $A \simeq (C_1 \to \dots \to C_n)$ (fin. gen. proj) ⇝ can reduce to the case $A = R$

Observe that everything is compatible ⇝ flat base change.

$\boxed{A = R[t]}$ omitted.

$\square$

Lemma. Let $f: R \to A$ be a regular closed immersion which is everywhere of pure codim c. (i.e. f surj. & locally $I = \ker(f)$ is gen. by a regular seq $f_1 \dots f_c$)

Then there ex. a nat. isom.

$$f^! R = R\mathrm{Hom}_R(A,R) \overset{!}{\cong} \left(\Lambda^c I/I^2\right)^\vee[-c]$$

Proof. Locally, $A = R/(f_1 \dots f_c)$ ⇝ $f_1 \dots f_c$ regular sequence.

Get

$$R\mathrm{Hom}_R(A,R) \underset{\substack{\uparrow \\ \text{Koszul computation}}}{\cong} A[-c] \xrightarrow{\cong} \left(\Lambda^c I/I^2\right)^\vee[-c]$$

$$1 \longmapsto (f_1 \wedge \dots \wedge f_c)^\vee$$

Can check that this compos. does not depend on choice of $f_1 \dots f_c$.

⇝ can glue the local isos.

$\square$

Thm A. Let $f: A \to R$ be smooth of rel. dim d.

Then $$f^! R \cong \left(\Lambda^d \Omega^1_{A/R}\right)[d]$$

Proof.
- have reg. closed imm $\mathrm{Spec}(A) \hookrightarrow \mathbb{A}^n_R$.
- Claim: $f^! R$ is line bundle concentrated in degree d.
  - affine space: $f^! R = R[t_1 \dots t_n][n]$ $(d = n - c)$
  - reg. closed imm $f^! R \underset{(\text{Lem 3})}{\cong} \left(\Lambda^c I/I^2\right)^\vee[-c]$

$$\begin{tikzcd}
\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} A \arrow[d, "p_1"'] \arrow[d, shift left=6, "p_2"] \\
\operatorname{Spec} A \arrow[u, shift left=3, "\Delta"]
\end{tikzcd}$$

Can compute:

$$f^! R = \Delta^! p_1^! f^! R \qquad (p_1^! \text{ is twist of } p_1^*)$$

$$= \Delta^! \Big(p_1^* f^! R \otimes^L_{(A \otimes_R A)} \underbrace{p_1^! A}_{= p_1^! f^* R} \Big)$$

$$\overset{(\text{Lem2 ii})}{=} \Delta^! \big(p_1^* f^! R \otimes^L_{(A \otimes_R A)} p_2^* f^* R \big)$$

$(\Delta^! \text{ is twist of } \Delta^*)$

$$= \Delta^* (\ldots) \otimes^L_{A} \Delta^! (A \otimes_R A)$$

$$= f^! R \otimes^L_{A} f^! R \otimes^L_{A} \Delta^! (A \otimes_R A)$$

$$\Rightarrow \quad f^! R \simeq \big(\Delta^! (A \otimes_R A) \big)^\vee \simeq \big(\Lambda^d I/I^2 \big)^\vee [-d]$$

w/ $\Omega^1_{A/R} \xrightarrow{\sim} I/I^2$

$\square$

Cor: For $X \xrightarrow{f} \operatorname{Spec} R$ as in Main Thm we have

$$f^! R \simeq \underbrace{\big(\Lambda^d \Omega^1_{X/R} \big)}_{= \omega_{X/R}} [d]$$

Proof of Main Thm [4] WTS:

$$\frac{f_! \, \omega_{X/R} [d] \xrightarrow{tr} R}{\omega_{X/R} [d] \xrightarrow{\sim} f^! R} \text{ adj.}$$

For $C \in D(\mathcal{O}_X)$ consider

$$\begin{tikzcd}
R\mathrm{Hom}_{\mathcal{O}_X}(C, \omega_{X/R})[d] \arrow[r] \arrow[dr, "\sim"'] & R\mathrm{Hom}_R(f_! C, R) \arrow[d, phantom, "\wr"] \\
& R\mathrm{Hom}_{\mathcal{O}_X}(C, f^! R)
\end{tikzcd}$$

$\square$