Globalization, II

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What was before

So far

- Introduced the derived category $\mathcal{D}((A, R)_{\blacksquare})$;
- Introduced the 6 operations for these categories.
- Introduced Adic spaces.

So now we want to consider the above constructions in the context of sheaves for $Spa(A, A^+)$.

What we want to do

The aim of this talk is to prove the following theorem.

Theorem 1

Let X be a discrete adic space. Then the functor $\mathcal{F}$, which associates to every adic open subset $Spa(A, A^+)$ the ∞-categors

$$\mathcal{D}((A, A^+)_{\blacksquare}) \subset \mathcal{D}(Cond(A))$$

is a sheaf on X with values in the ∞-category of ∞-categories.

So how does one go about proving this theorem?

"Sketch" of proof

Let $X = \bigcup_{i=1}^{n} U_i$ be a cover with $U_i = Spa(A_i, A_i^+)$ for some discrete Huber pair (A_i, A_i^+). Consider $I \subset \{1, .., n\}$. Then the intersection $U_I = \bigcap_{i \in I} U_i$ is of the form $Spa(A_I, A_I^+)$ for some (A_I, A_I^+).
Let $Cond_{U_I}(O_X)$ be the category of presheaves of condensed O_X modules which are defined only on the $\{U_I\}$, which is an abelian category with compact projective generators. Then

$$\mathcal{D}((O_X, O_X^+)_\blacksquare) \to \mathcal{D}(Cond_{U_I}(O_X))$$

is fully faithful with essential image consisting of $M \in \mathcal{D}(Cond_{U_I}(O_X))$ with $M_I \in \mathcal{D}(Cond(A_I))$ in the essential image of $\mathcal{D}((A_I, A_I^+)_\blacksquare)$. The base-change maps

$$M_I \otimes^L_{(A_I, A_I^+)} (A_J, A_J^+)_\blacksquare \to M_J$$

are equivalences. So we're done, right?

Outline

One has to show that the construction is independent of the choice of cover of X. To this end we will proceed in two steps:

1. Prove a very general theorem about sheaves on spectral spaces with values in ∞-categories.
2. Check that this theorem applies to our setting.

Spectral spaces

Reminder: A topological space X is spectral, if it's homeomorphic to $Spec(A)$ of some ring A. The following theorem is due to Hochster.

Theorem 2

The following statements are equivalent.

1. X *is a spectral space*
2. X *is quasi-compact, has a basis of quasi-compact open sets, stable under intersection and sober, i.e. every irreducible component has a unique generic point.*
3. X *is an inverse limit of finite* T_0 *spaces.*

"A very general" theorem

Theorem 3

Let X a spectral space with a basis B of quasi-compact open subsets, stable under intersection. Let C be a stable ∞-category. Let $U \to C_U \subset C$ be a covariant functor from B to the full subcategories of C, such that the inclusion $C_U \to C$ admits a left adjoint L_U. Let C_U also satisfy the following

- *If U, V are open sets in the basis B, then $C_{U \cap V} = C_U \cap C_V$ and $L_{U \cap V} = L_U \circ L_V = L_V \circ L_U$;*
- *If U is covered by $U_1, ..U_n \in B$ and $M \in C_U$ with $L_{U_i}(M) = 0$ for all i, then $M = 0$.*

Then this functor defines a sheaf of ∞-categories and it admits a left adjoint L_U.

Proof of the "very general" theorem

Let U be an open set in the basis B and let $\{U_i\}_{1\leq i\leq n}$ be a finite cover by open sets in B. Let I be a subset of $1,..,n$ and consider the category $C_{U_I}\subset C_U$ with $U_I=\bigcap_{i\in I} U_i$. Then $C_I\to C_U$ admits a left adjoint $L_I: C_U\to C_I$. Then for all I the functors L_{U_I} define a natural functor

$$F=(L_I): C_U\to \varprojlim_I C_I.$$

Claim: F is an equivalence of categories.
Proof: Since C_U also satisfies the assumptions of the theorem, we may assume that $U=X$. Let M be an object in C. Then we need to show that $M\to \varprojlim_I L_I(M)$ is an equivalence. Let $J_i\subset\{1,..,n\}$ be a finite set, such that $U_j\cap U_i\neq\emptyset$ for all $j\in J_i$.

Proof of the "very general" theorem

Consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{(L_{U_i})} & \coprod_{1\leq i\leq n} C_{U_i} \\ {\scriptstyle (L_I)}\big\downarrow & & \big\downarrow{\scriptstyle (L_{J_i})} \\ \varprojlim_I C_I & \xrightarrow{(L_{U_i})} & \coprod_{1\leq i\leq n} \varprojlim_{J_i} (C_{U_i})_{J_i} \end{array}$$

which is commutative, since the inverse limits are finite. The horizontal arrows are inclusions due to condition 2. Then by condition 1, it suffices to prove the statement for the case when $U_i = X$ for some i. But then the cover is split, so it's automatically an equivalence.

Reminder

Recall, that given an adic space $Spa(A, A^+)$, then a rational subset U is a subset of the form

$$U(\frac{g_1, \dots, g_n}{f}) = \{| \ | \in Spa(A, A^+) \ || \ g_i \ | \leq | \ f \ | \ \text{ for all } i \text{ and } \ | \ f \ | \neq 0\}.$$

Also recall prop. 9.3 from the notes / previous talk

Prop. 9.3

Let $(A, A^+) \to (B, B^+)$ be a map of discrete Huber pairs, such that the induced map between the adic spaces factors through a rational subset $U = U(\frac{g_1, \dots, g_n}{f})$. Then the map factors uniquely over the pair $(A[\frac{1}{f}], A^+[\widetilde{\frac{g_1}{f}, \dots,} \frac{g_n}{f}])$ and $Spa((A[\frac{1}{f}], A^+[\widetilde{\frac{g_1}{f}, \dots,} \frac{g_n}{f}]))$ is homeomorphic to U.

We will use this routinely to reduce the proofs to very simple cases of rational subsets.

Localization for $\mathcal{D}(\mathcal{A}, \mathcal{A}^+)$

Theorem 4

Let (A, A^+) be a discrete Huber pair and let $U \subset X = Spa(A, A^+)$ be a rational subset. Then the forgetful functor

$$\mathcal{D}((O_X(U), O_X^+(U))_\blacksquare) \to \mathcal{D}((A, A^+)_\blacksquare)$$

is fully faithful and admits a left adjoint $\otimes^L_{(A,A^+)_\blacksquare}(O_X(U), O_X^+(U))_\blacksquare$.

Proof: Consider the diagram

$$\begin{array}{ccc} \mathcal{D}((O_X(U), O_X^+(U))_\blacksquare) & \longrightarrow & \mathcal{D}((A, A^+)_\blacksquare) \\ \downarrow & & \downarrow \\ \mathcal{D}(Cond(O_X(U))) & \longrightarrow & \mathcal{D}(Cond(A)) \end{array}$$

where horizontal arrows are forgetful functors. By Prop. 9.3 in the notes, we may assume that $U = U(\frac{1}{f})$ for some $f \in A$. Then the result follows from full faithfulness of the bottom Horizontal arrow. $\square$

Localization and base-change

Theorem 5

Let $(A, A^+) \to (B, B^+)$ be a map of discrete Huber pairs. Denote $X = Spa(A, A^+)$ and let $U \subset X$ be a rational subset with preimage $V \subset Y = Spa(B, B^+)$ and consider the diagram of forgetful functors

$$\begin{array}{ccc} \mathcal{D}((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare) & \longrightarrow & \mathcal{D}((B, B^+)_\blacksquare) \\ \downarrow & & \downarrow \\ \mathcal{D}((\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))_\blacksquare) & \longrightarrow & \mathcal{D}((\mathcal{A}, \mathcal{A}^+)_\blacksquare). \end{array}$$

This diagram is Cartesian and the following diagram induced by adjoint functors also commutes

Theorem 5

$$\begin{array}{ccc}
\mathcal{D}((B,B^+)_{\blacksquare}) & \xrightarrow{-\otimes^L_{(B,B^+)_{\blacksquare}}(\mathcal{O}_X(U),\mathcal{O}_X^+(U))_{\blacksquare}} & \mathcal{D}((\mathcal{O}_X(U),\mathcal{O}_X^+(U))_{\blacksquare}) \\
\downarrow & & \downarrow \\
\mathcal{D}((\mathcal{A},\mathcal{A}^+)_{\blacksquare}) & \xrightarrow{-\otimes^L_{(A,A^+)_{\blacksquare}}(\mathcal{O}_Y(V),\mathcal{O}_Y^+(V))_{\blacksquare}} & \mathcal{D}((\mathcal{O}_Y(V),\mathcal{O}_Y^+(V))_{\blacksquare}).
\end{array}$$

Localization and base-change

Proof: Let $U = U(\frac{g_1,\dots,g_n}{f})$. Then $U = \bigcap_{i=1}^{n} U(\frac{g_i}{f})$, so by induction it suffices to prove for the case $U = (\frac{g}{f})$. Either $f = g$ or $f \neq g$ so by multiplicativity of valuations we need to consider only the cases $U = U(\frac{g}{1})$ and $U = U(\frac{f}{f})$.
Case $U = U(\frac{f}{f})$: We may assume that $U = U(\frac{1}{f})$ for some $f \in A$ and A finitely generated over $\mathbb{Z}$. In this case

$$(O_X(U)(f), O_X^+(U))_{\blacksquare} = (A[\tfrac{1}{f}], A^+)_{\blacksquare} = (A, A^+)_{\blacksquare} \otimes_A A[\tfrac{1}{f}]$$

since $(A, A^+)_{\blacksquare}$ is the condensed ring associated with the functor $S \to A^+_{\blacksquare}[S] \otimes_{A^+} A$. This implies, that

$$\mathcal{D}((O_X(U), O_X^+(U))_{\blacksquare}) \subset \mathcal{D}((A, A^+)_{\blacksquare})$$

is the full subcategory of $M \in \mathcal{D}((A, A^+)_{\blacksquare})$, on which multiplication by f is invertible. And the adjoint functor is given by $M \to M\frac{1}{f}$, which is compatible with base change.

Localization and base-change

Case $U = U(\frac{g}{1})$: We may assume, that $(A, A^+) = (\mathbb{Z}[T], \mathbb{Z})$ with $g = T$. In that case $(O_X(U), O_X^+(U)) = (A, A)$ and by Tony's talk $- \otimes^L_{(A,A^+)_\blacksquare} (A, A)$ is given by

$$R\underline{\mathrm{Hom}}_A(A_\infty/A, -)[1]$$

where $A_\infty = \mathbb{Z}((T^{-1}))$. Then the result follows, if we show, that for any $M \in \mathcal{D}((B, B^+)_\blacksquare)$ the base change $M \otimes^L_{(A,A)_\blacksquare} (A, A)_\blacksquare$ lies in $\mathcal{D}((B, \widetilde{B^+[T]}))$. We may assume, that B is of finite type and $B^+[T]$ is a polynomial algebra over B^+. Then since $B_\infty/B = A_\infty/A \otimes_{\mathbb{Z}} B^+_\blacksquare$ and the adjoint functor is given by $R\underline{\mathrm{Hom}}_B(B_\infty/B, -)[1]$, it follows that

$$M \otimes^L_{(A,A^+)_\blacksquare} (A, A)_\blacksquare = M \otimes^L_{(B,B^+)_\blacksquare} (B, B)_\blacksquare$$

□

Verifying the conditions

Now we need to verify that the pre-sheaf in theorem 1 satisfies assumptions 1 and 2 of the very general theorem. To this end we will introduce 2 lemmas. The proof of the second requires the first one.

Lemma 6

Let (A, A^+) be a discrete Huber pair and $X = Spa(A, A^+)$. Let $\{U_i\}_{1 \leq i \leq n}$ be a cover of rational subsets of X. Then there exist $s_1, ..., s_m \in A$, such that they generate A and each $U(\frac{s_1 \dots s_m}{s_j})$ is contained in some U_i.

Lemma 7

1. *Let U, V be rational subsets. Then $C_{U \cap V} = C_U \cap C_V$ and $L_{U \cap V} = L_U \circ L_V = L_V \circ L_U$.*
2. *Let $\{U_i\}_{1 \leq i \leq n}$ be a cover of rational subsets of X. If for $M \in C$ and all i we have $L_i(M) = 0$, then $M = 0$.*

Proof of lemma 6

Part (I): Let $U \subset V \subset X$ be open subsets and let S denote a pro-finite set. Recall that given a map of rings $f : R \to A$, the pre-analytic ring $(A, R)_\blacksquare$ is given by the condensed ring A and the functor $S \to R_\blacksquare[S] \otimes_R A$. Then we have natural maps

$$(A, A^+)_\blacksquare \to (\mathcal{O}_X(V), \mathcal{O}_X^+(V))_\blacksquare \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare$$

which show on one hand $\mathcal{C}_U \subset \mathcal{C}_V \subset \mathcal{C}$ and on the other hand induce forgetful functors. Then $\mathcal{C}_{U \cap V}$ is contained in $\mathcal{C}_U \cap \mathcal{C}_V$. For the reverse inclusion, let $M \in \mathcal{C}_U \cap \mathcal{C}_V$. Then indeed $M = L_U \circ L_V(M) = L_{U \cap V}(M)$ and the statement follows from compatibility of localization with base-change, i.e. theorem 5.

Proof of lemma 6

Part (II): The proof follows by induction. Recall that given a Huber pair (A, A^+), the rational subset $U(\frac{f}{g}) \subset Spa(A, A^+)$ consists of those valuations $|\ |$ in $Spa(A, A^+)$, that $| g | \geq | f |$.
Let $n = 2$ and let $U_1 = U(\frac{1}{f})$ and $U_2 = U(\frac{f}{1})$. Then U_1 and U_2 form a cover of X. In this case by theorem 5 we may assume, that $A = \mathbb{Z}[T]$ and $f = T$. Since $L_{U_1}(M) = 0$ iff M is an $\mathbb{Z}[[T]]$ module and $L_{U_2}(M) = 0$ iff M is a $\mathbb{Z}((T^{-1}))$ module. Then M is a $\mathbb{Z}[[T]] \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\blacksquare} \mathbb{Z}((T^{-1}))$-module and this ring is 0.

Proof of lemma 6

Finally by using lemma 6 assume, that $U_i = U(\frac{f_1 \dots f_n}{f_i})$, where $f_1, .., f_n$ generate A. Since $U_i = \bigcap_{j=1}^{n} U(\frac{f_j}{f_i})$, it is enough to show, that $M[\frac{1}{f_i}] = 0$ for all $i = 1, .., n$. We may also assume, that one of these f_i, say f_n is equal to 1. Then by induction hypothesis we need to show that localization of M to $U(\frac{1}{f_1})$ and $U(\frac{f_1}{1})$ is 0, so f_1 or f_1^{-1} is in A^+. In the first case

$$U_i = U(\frac{f_1, \dots, f_{n-1}, 1}{f_i}) = U(\frac{f_2, \dots, f_{n-1}, 1}{f_i})$$

and for the second case

$$U_i = U(\frac{f_1, \dots, f_{n-1}, 1}{f_i}) = U(\frac{1, f_1^{-1} f_2, \dots, f_1^{-1} f_{n-1}}{f_1^{-1} f_i}).$$

This reduces to a cover consisting of $n-1$ rational subsets.

Questions?

Any questions?

Thank you

Thanks for listening!