$S \in \mathrm{Prof}$, $S = \lim_i S_i$

$$\mathbb{Z}[S] \longrightarrow \lim_i \mathbb{Z}[S_i] := \mathbb{Z}[S]^{\blacksquare} \in \mathrm{Cond}(\mathrm{Ab})$$

<u>Def</u> • A condensed abelian group A is <u>solid</u> if $\forall S \in \mathrm{Prof}$

$$\mathrm{Hom}(\mathbb{Z}[S]^{\square}, A) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[S], A)$$

• A complex $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ is <u>solid</u> if $\forall S \in \mathrm{Prof}$

$$R\mathrm{Hom}(\mathbb{Z}[S]^{\square}, C) \xrightarrow{\sim} R\mathrm{Hom}(\mathbb{Z}[S], C).$$

<u>Goal</u>: $\mathbb{Z}[S]^{\square}$ is solid.

<u>Examples</u>:
- discrete ab. groups, $\mathbb{Z}_p$, $\mathbb{Z}[\![t]\!]$ are solid
- $\mathbb{R}$ is not solid. $\mathbb{R}^{\square} = 0$
- $(\mathbb{Z}_p \otimes \mathbb{Z}[\![t]\!])^{\square} = \mathbb{Z}_p[\![t]\!]$
- X is a CW complex $\quad \mathbb{Z}[X]^{L\square} = C_*(X) \in D(\mathrm{Ab})$.

$$\mathbb{Z}[S]^{\square} = \lim_i \mathbb{Z}[S_i] = \lim_i \underline{\mathrm{Hom}}(C(S_i,\mathbb{Z}),\mathbb{Z}) = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$$

$$C(S,\mathbb{Z}) = \{\text{continuous maps } S \to \mathbb{Z}\} = \operatorname{colim}_i C(S_i,\mathbb{Z})$$

$\Rightarrow$ the underlying ab. group of $\mathbb{Z}[S]^{\square}$ is $\mathrm{Hom}(C(S,\mathbb{Z}),\mathbb{Z})$

<u>Theorem</u> (Nöbeling) $C(S,\mathbb{Z})$ is a free abelian group, $\forall S \in \mathrm{Prof}$.

<u>Corollary</u> $\mathbb{Z}[S]^{\square} = \prod_I \mathbb{Z}$ in $\mathrm{Cond}(\mathrm{Ab})$

<u>Corollary</u> If $S_\bullet \to S$ is a hypercover in Prof, then

$$\cdots \to \mathbb{Z}[S_1]^{\square} \to \mathbb{Z}[S_0]^{\square} \to \mathbb{Z}[S]^{\square} \to 0$$

is exact. Hence, in the definition of solid, we can assume S extr. disconnected.

<u>Pf.</u> $H^i(S,\mathbb{Z}) = 0$ for $i > 0$

$$\Rightarrow \quad 0 \to C(S,\mathbb{Z}) \to C(S_0,\mathbb{Z}) \to C(S_1,\mathbb{Z}) \to \cdots \quad \text{exact}$$

Apply $\underline{\mathrm{Hom}}(-,\mathbb{Z}) \underset{(AB4^*)}{=} R\underline{\mathrm{Hom}}(-,\mathbb{Z})$ ∎

<u>Proposition</u> $\mathbb{Z}[S]^{\square}$ is solid in $\mathrm{Cond}(\mathrm{Ab})$ and $D(\mathrm{Cond}(\mathrm{Ab}))$.

Let $T \in \mathrm{Prof}$. We need $R\mathrm{Hom}(\mathbb{Z}[T]^{\square}, \mathbb{Z}[S]^{\square}) \xrightarrow{\sim} R\mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}[S]^{\square})$

$\mathbb{Z}[S]^\square = \prod_I \mathbb{Z} \quad \Rightarrow \quad$ WLOG $\mathbb{Z}[S]^\square = \mathbb{Z}$

Also, $\mathbb{Z}[T]^\square = \prod_\gamma \mathbb{Z}$.

<u>RHS</u>: $\quad R\mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}) = C(T, \mathbb{Z}) = \bigoplus_\gamma \mathbb{Z}$ in degree 0

<u>LHS</u>: We use the exact sequence

$$0 \to \underset{\mathbb{Z}[T]^\square}{\underset{\parallel}{\prod_\gamma \mathbb{Z}}} \to \prod_\gamma \mathbb{R} \to \prod_\gamma \mathbb{R}/\mathbb{Z} \to 0$$

$\Rightarrow$ fiber sequence

$$R\mathrm{Hom}(\mathbb{Z}[T]^\square, \mathbb{Z}) \leftarrow \underset{0}{\underset{\parallel}{R\mathrm{Hom}\big(\prod_\gamma \mathbb{R}, \mathbb{Z}\big)}} \leftarrow \underset{\bigoplus_\gamma \mathbb{Z}[-1]}{\underset{\parallel}{R\mathrm{Hom}\big(\prod_\gamma \mathbb{R}/\mathbb{Z}, \mathbb{Z}\big)}}$$

By the key computation from last time:

$$R\mathrm{Hom}\big(\prod_\gamma \mathbb{R}, \mathbb{Z}\big) = R\mathrm{Hom}_{\mathbb{R}}\big(\prod_\gamma \mathbb{R}, \underbrace{R\mathrm{Hom}(\mathbb{R}, \mathbb{Z})}_{=0}\big)$$

$$\underset{\mathbb{Z}[-1]}{\underset{\parallel}{R\mathrm{Hom}(\mathbb{R}/\mathbb{Z}, \mathbb{Z})}} \to \underset{0}{\underset{\parallel}{R\mathrm{Hom}(\mathbb{R}, \mathbb{Z})}} \to \underset{\mathbb{Z}}{\underset{\parallel}{R\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}}$$

$\underline{\partial = \mathrm{id}}$

$\Rightarrow \quad R\mathrm{Hom}(\mathbb{Z}[T]^\square, \mathbb{Z}) = \bigoplus_\gamma \mathbb{Z}$. $\quad \blacksquare$

Nöbeling's theorem follows from:

<u>Lemma</u> Let R be a commutative ring generated by idempotents and torsion-free. Then $(R, +)$ is a free abelian group.

<u>Proof</u>. Choose an ordinal τ and a sequence $(\varepsilon_\sigma)_{\sigma < \tau}$ of idempotents generating R. Put the lexicographical ordering on the set of sequences $\sigma_1 > \sigma_2 > \dots > \sigma_k$

$$(\emptyset < 0 < 1 < 10 < 2 < 20 < 21 < 210 < \dots)$$

<u>Claim</u> The products $\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_k}$, $\sigma_1 > \dots > \sigma_k$, that are not $\mathbb{Z}$-linear combinations of smaller products form a basis of $(R, +)$.

Induction on τ:

- $\tau = 0$: $R = \mathbb{Z}$ or $R = 0$ ✓
- τ limit ordinal : easy
- $\tau = \rho + 1$. Let $\varepsilon = \varepsilon_\rho$, let $R_0 \subset R$ subring generated by ε_σ, $\sigma < \rho$.

Note : $\varepsilon R \to R/R_0$ surjective

$$\Rightarrow R/R_0 = \varepsilon R / \underset{\text{ideal in } \varepsilon R}{(R_0 \cap \varepsilon R)} = \overline{R}$$

R_0 is generated by $(\varepsilon_\sigma)_{\sigma<\rho}$ and is torsion-free

$\overline{R}$ is generated by $(\overline{\varepsilon\varepsilon_\sigma})_{\sigma<\rho}$. Claim : $\overline{R}$ is torsion-free.

$\Rightarrow$ We can apply induction hyp to R_0 and $\overline{R}$:

In R_0 : products $\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_k}$ with $\sigma_i < \rho$ that are not linear combinations of smaller products form a basis of R_0

In $\overline{R}$: products $\overline{\varepsilon\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_k}}$, $\sigma_i < \rho$, ——————— form a basis of $\overline{R}$.

The corresponding products $\varepsilon\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_k}$ in R are precisely the elements of the candidate basis that start with ε.

$$\begin{array}{ccccc} & R_0 & \hookrightarrow & R & \twoheadrightarrow \overline{R} \\ & \cup & & & \\ \text{basis} & E_0 & \hookrightarrow & E_0 \cup E' & \end{array}$$

$E_0 \longmapsto 0$

$E' \xrightarrow{\sim} \text{basis} \quad \Longrightarrow \quad E_0 \cup E'$ basis of R.

$\overline{R}$ torsion-free $\Longleftrightarrow$ $R_0/p \to R/p$ injective $\forall$ prime p

$R_0 \subset R$ integral ext. $\Rightarrow$ $\mathrm{Spec}\, R/p \twoheadrightarrow \mathrm{Spec}\, R_0/p$

$\Rightarrow \ker(R_0/p \to R/p)$ is nilpotent.

but R_0/p has no nilpotents (every element in R_0 is a linear combination of orthogonal idempotents) ☒

<u>Theorem</u> (next time)

- Solid $\subset$ Cond(Ab) is abelian subcategory closed under limits, colimits, extensions
- $\prod_I \mathbb{Z}$, I any set, family of compact projective generators of Solid.
- $D(\text{Solid}) \to D(\text{Cond(Ab)})$ is fully faithful with ess. image the solid complexes
- $C \in D(\text{Cond(Ab)})$ is solid $\Longleftarrow$ $H^i(C) \in$ Solid for all $i \in \mathbb{Z}$.

Proposition Let A be a cocomplete abelian category, $A_0 \subset A$ a subcategory of compact projective generators, closed under $\oplus$. $\left(\Rightarrow \begin{array}{l} D(A)_{\geq 0} = P_\Sigma(A_0) \\ A = P_\Sigma(A_0)_{\leq 0} \end{array} \right)$

Let $F : A_0 \to A$ be an additive functor with a natural transformation

$P \longrightarrow F(P)$, $P \in A_0$.

Let $A_F \subset A$ subcategory of all X s.t.:

$$Hom(F(P), X) \xrightarrow{\sim} Hom(P, X) \quad \forall P \in A_0$$

Let $D_F(A) \subset D(A)$ subcategory of all C s.t.

$$RHom(F(P), C) \xrightarrow{\sim} RHom(P, C) \quad \forall P \in A_0$$

Suppose:

(*) Every complex

$$C = \cdots \to C_1 \to C_0 \to 0$$

where each C_i is a sum of objects in the range of F belongs to $D_F(A)$.

Then: (i) $A_F \subset A$ is an abelian subcategory closed under limits, colimits, extensions. The objects $F(P)$, $P \in A_0$, are a family of compact generators of A_F, and the left adjoint $L : A \to A_F$ is the LKE of F.

(ii) $D(A_F) \to D(A)$ is fully faithful with ess. range $D_F(A)$. $C \in D(A)$ belongs to $D_F(A)$ iff $H_i(C) \in A_F$ for all i. The left adjoint $D(A) \to D_F(A)$ is the left derived functor of L.

Proof Step 1 $F(A_0) \subset A_F$. $P \in A_0$

Apply (*) with $C = F(P)[0]$

$$\Rightarrow \forall Q \in A_0: \quad \underset{\substack{\| \\ Hom(F(Q), F(P))}}{RHom(F(Q), C)} \xrightarrow{\sim} RHom(Q, C) = Hom(Q, F(P))$$

$$\Rightarrow F(P) \in A_F.$$

Step 2 $A_F = A \cap D_F(A)$

$\supset$: obvious

$\mathcal{A}_F \subset D_F(\mathcal{A})$. Let $X \in \mathcal{A}_F$. Choose a resolution

$$P_* \longrightarrow X \quad \text{where} \quad P_i = \bigoplus_j P_{i,j} \quad P_{i,j} \in \mathcal{A}_0$$

Let $Q_* = $ "$F(P_*)$" $\quad Q_i = \bigoplus_j Q_{ij} \quad Q_{i,j} := F(P_{ij})$

Since $X \in \mathcal{A}_F$, we have a factorization

$$\begin{tikzcd} P_* \arrow[d] \arrow[dr, "\sim"] & \\ Q_* \arrow[r] & X. \end{tikzcd}$$

$\Rightarrow$ X is a retract of Q_* in $D(\mathcal{A})$

But $Q_* \in D_F(\mathcal{A})$ by $(*)$ $\Rightarrow$ $X \in D_F(\mathcal{A})$.

Step 1 $\Rightarrow$ $F(\mathcal{A}_0)$ are compact projective generators of $\mathcal{A}_F$

$$\mathcal{A} = P_\Sigma(\mathcal{A}_0)_{\leq 0} \underset{\text{res}}{\overset{\text{LKE}}{\rightleftarrows}} \mathcal{A}_F = P_\Sigma(F(\mathcal{A}_0))_{\leq 0}$$

$\Rightarrow$ all of (i) except extensions

Step 2 $\Rightarrow$ $\mathcal{A}_F \subset \mathcal{A}$ closed under extensions.

Step 3. $D(\mathcal{A}_F) \to D(\mathcal{A})$ is fully faithful. $P \in \mathcal{A}_0$, $C \in Ch(\mathcal{A}_F)$

$$\begin{array}{ccc} R\mathrm{Hom}_{D(\mathcal{A}_F)}(F(P), C) & \longrightarrow & R\mathrm{Hom}_{D(\mathcal{A})}(F(P), C) \\ \uparrow & & \uparrow \\ R\mathrm{Hom}_{D(\mathcal{A}_F)}(F(P), \tau_{\geq i} C) & \longrightarrow & R\mathrm{Hom}_{D(\mathcal{A})}(F(P), \tau_{\geq i} C) \end{array}$$

iso on H_j for $j \geq i$

WLOG C is bounded below.

$$R\mathrm{Hom}_{D(\mathcal{A}_F)}(-, C) = \lim_i R\mathrm{Hom}_{D(\mathcal{A}_F)}(-, \tau_{\geq i} C) \quad \text{same in } D(\mathcal{A})$$

$\Rightarrow$ WLOG $C = X[0]$, $X \in \mathcal{A}_F \subset D_F(\mathcal{A})$

$$R\mathrm{Hom}_{D(\mathcal{A}_F)}(F(P), C) = \mathrm{Hom}(F(P), X) \quad X \in \mathcal{A}_F.$$

$$R\mathrm{Hom}_{D(\mathcal{A})}(F(P), C) \underset{C \in D_F(\mathcal{A})}{=} R\mathrm{Hom}_{D(\mathcal{A})}(P, C) = \mathrm{Hom}(P, X)$$

$\blacksquare$