Chapter 3: Cohomology

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Notation

Throughout this talk, denote:

$S \in \mathbf{Comp}$ a quasi-compact Hausdorff (*compact*) topological space.

$\mathrm{Cond}(\mathbf{Set}), \mathrm{Cond}(\mathbf{Ab})$ the categories of condensed sets/abelian groups.

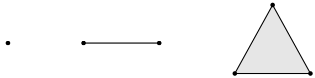
Aim: Discuss notions of $\mathrm{H}^\bullet(S, A)$ for A an abelian group.

Singular cohomology

Start with the space S.

Consider the simplicial set

$\mathscr{S}_n = \mathrm{Hom}_{\mathbf{Top}}(n\text{-Simplex}, S).$



Turn into a chain complex

$$C_\bullet : \qquad \cdots \to \mathbb{Z}[\mathscr{S}_2] \to \mathbb{Z}[\mathscr{S}_1] \to \mathbb{Z}[\mathscr{S}_0] \to 0,$$

where d_i is the alternating sum of the $i+1$ face maps.
Then $\mathrm{H}^\bullet_{\mathrm{sing}}(S, A) =$ cohomology of $\mathrm{Hom}_{\mathbf{Ab}}(C_\bullet, A)$.

Čech cohomology

Turn A into a constant sheaf on S:
$\Gamma(U, A) = \mathrm{Hom}_{\mathbf{Top}}(U, A^{\mathrm{discrete}})$ for all U.

For a finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ on S, form a cosimplicial space

$$S_0 := \coprod \mathcal{U}, \quad S_n = \underbrace{S_0 \underset{S}{\times} \cdots \underset{S}{\times} S_0}_{n+1 \text{ times}}.$$

The alternating sum of the face (projection) maps $S_n \to S_{n-1}$ give

$$0 \to \Gamma(S_0, A) \to \Gamma(S_1, A) \to \Gamma(S_2, A) \to \cdots$$

$\mathrm{H}^\bullet$ of this complex is $\mathrm{H}^\bullet_{\text{Čech}}(\mathcal{U}, A)$. $\mathrm{H}^\bullet_{\text{Čech}}(S, A) = \varinjlim_{\mathcal{U}} \mathrm{H}^\bullet_{\text{Čech}}(\mathcal{U}, A)$.

Sheaf cohomology

The functor

$$\text{(abelian sheaves over } S) \xrightarrow{\Gamma} \mathbf{Ab}$$

has right-derived functors.

$\mathrm{H}^{\bullet}_{\text{sheaf}}(S, A) = R^{\bullet}\Gamma(S, A)$.

Compute e.g. using injective resolution $A \to I^{\bullet}$, then $\mathrm{H}^{\bullet}_{\text{sheaf}}(S, A) =$ cohomology of $\Gamma(S, I^{\bullet})$.

Comparison

Lemma.

$\mathrm{H}^\bullet_{\check{\mathrm{C}}\mathrm{ech}}(S, A) \cong \mathrm{H}^\bullet_{\mathrm{sheaf}}(S, A)$.

If S is a profinite set and A discrete,

$$\mathrm{H}^0_{\check{\mathrm{C}}\mathrm{ech}}(S, A) \cong \mathrm{H}^0_{\mathrm{sheaf}}(S, A) \cong \mathrm{Hom}_{\mathbf{Top}}(S, A).$$
$$\mathrm{H}^0_{\mathrm{sing}}(S, A) \cong \mathrm{Hom}_{\mathbf{Set}}(S, A).$$

Sheaf (Čech) cohomology is better suited for condensed mathematics.

1. Different notions of cohomology
   - Cohomology from topology
   - Cohomology from the topos
2. Cohomology with integral coefficients
3. Cohomology with real coefficients

Condensed cohomology

Recall: $\mathrm{Cond}(\mathbf{Ab}) \simeq$ category of abelian sheaves over **Comp**.

Definition.

We define $\mathrm{H}^{\bullet}_{\mathrm{cond}}(S, \cdot) : \mathrm{Cond}(\mathbf{Ab}) \to \mathbf{Ab}$ to be the right-derived functors of $\Gamma(S, \cdot)$.

Since $\Gamma(S, A) \cong \mathrm{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[S], A)$, conclude

$$\mathrm{H}^{\bullet}_{\mathrm{cond}}(S, A) \cong \mathrm{Ext}^{\bullet}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[S], A).$$

May use a projective resolution of $\mathbb{Z}[S]$ or injective resolution of A.

Resolving S

If S is extremally disconnected (projective), then
$\Gamma(S,\cdot) : \mathrm{Cond}(\mathbf{Ab}) \to \mathbf{Ab}$ is exact $\Longrightarrow \mathrm{H}^{\geq 1}_{\mathrm{cond}}(S,\cdot) = 0$.

In general, we want a "resolution" in **Comp**

$$S_\bullet = (S_n)_{n\geq 0} \ + \ \text{simplicial structure}$$

with each S_n extremally disconnected.
These should give rise to a projective resolution

$$\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[S] \to 0$$

in $\mathrm{Cond}(\mathbf{Ab})$.

As $\mathrm{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[S_n],\cdot) \cong \Gamma(S_n,\cdot)$ is exact, $\mathbb{Z}[S_n]$ is projective in $\mathrm{Cond}(\mathbf{Ab})$.

Let's try Čech

Pick $S_0 \to S$ surjective such that S_0 is extremally disconnected (e.g. Stone-Čech compactification of S^{discrete}).

For $n \geq 1$, let $S_n = \underbrace{S_0 \times_S \cdots \times_S S_0}_{n+1 \text{ times}}$.

Usual arguments show the Čech complex

$$\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[S] \to 0$$

is exact!

Problem: S_n not necessarily extremally disconnected for $n \geq 1$.

Hypercover

Need something better:
Pick $S_0 \to S$ surjective with S_0 extremally disconnected.
Pick $S_1 \twoheadrightarrow S_0 \times_S S_0$ with S_1 extremally disconnected.

Pick

$$S_2 \twoheadrightarrow \left\{ (u,v,w) \in S_1 \times S_1 \times S_1 \mid \begin{array}{c} d_1(u) = d_1(v), \\ d_2(u) = d_1(w), \\ d_2(v) = d_2(w) \end{array} \right\},$$

where $d_{1,2}$ is $S_1 \to S_0 \times_S S_0 \xrightarrow{\pi_{1,2}} S_0$.

Generally pick $S_{n+1} \twoheadrightarrow$ Coskeleton at level $n+1$ of the truncated simplical set $S_0, \ldots, S_n$.

Computing $\mathrm{H}^\bullet_{\mathrm{cond}}(S, A)$

Pick a hypercover $S_\bullet \to S$ such that each S_n is extremally disconnected (at least, $\mathbb{Z}[S_n]$ is $\mathrm{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\cdot, A)$-acyclic).

This yields a projective (acyclic) resolution

$$\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[S] \to 0.$$

Then $\mathrm{H}^\bullet_{\mathrm{cond}}(S, A) = \mathrm{Ext}^\bullet(\mathbb{Z}[S], A)$ is the cohomology of

$$\begin{aligned} &0 \to \mathrm{Hom}_{\mathrm{Cond}(\mathbf{Ab})}(\mathbb{Z}[S_0], A) \to \mathrm{Hom}(\mathbb{Z}[S_1], A) \to \mathrm{Hom}(\mathbb{Z}[S_2], A) \cdots \\ =&0 \to \Gamma(S_0, A) \to \Gamma(S_1, A) \to \Gamma(S_2, A) \cdots , \end{aligned}$$

where each codifferential is the alternating sum of face maps.

The result

Theorem (Dyckhoff, 1976).

There is an isomorphism

$$\mathrm{H}^\bullet_{\mathsf{cond}}(S, \mathbb{Z}) \cong \mathrm{H}^\bullet_{\mathsf{sheaf}}(S, \mathbb{Z})$$

which is natural in S.

Observe that in particular the constant sheaf $\mathbb{Z} \in \mathsf{Cond}(\mathbf{Ab})$ has infinite injective dimension (unlike $\mathbb{Z} \in \mathbf{Ab}$).

The proof should work for any discrete abelian group.

If S is a finite set,

$$\mathrm{H}^n_{\mathsf{cond}}(S, \mathbb{Z}) = \mathrm{H}^n_{\mathsf{sheaf}}(S, \mathbb{Z}) = \begin{cases} \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{Z}), & n = 0 \\ 0, & n \geq 1 \end{cases}.$$

1. Different notions of cohomology

2. Cohomology with integral coefficients
   - Case of profinite sets
   - General case

3. Cohomology with real coefficients

Computing $\mathrm{H}^\bullet_{\mathrm{sheaf}}(S, \mathbb{Z})$

Let $S = \varprojlim_j S^j$ be profinite.
Then

$$\mathrm{H}^\bullet_{\mathrm{sheaf}}(S, \mathbb{Z}) \cong \mathrm{H}^\bullet_{\check{\mathrm{C}}\mathrm{ech}}(S, \mathbb{Z}) \cong \varinjlim_j \mathrm{H}^\bullet_{\check{\mathrm{C}}\mathrm{ech}}(S^j, \mathbb{Z}).$$

Eilenberg-Steenrod: *Foundations of algebraic topology*, Chapter X, Theorem 3.1

Now $\mathrm{H}^{\geq 1}_{\check{\mathrm{C}}\mathrm{ech}}(S^j, \mathbb{Z}) = 0$. Thus $\mathrm{H}^{\geq 1}_{\mathrm{sheaf}}(S, \mathbb{Z}) = 0$ and

$$\mathrm{H}^0_{\mathrm{sheaf}}(S, \mathbb{Z}) = \Gamma(S, \mathbb{Z}) = \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{Z}).$$

Computing $\mathrm{H}^\bullet_{\mathrm{cond}}(S, \mathbb{Z})$

Certainly $\mathrm{H}^0_{\mathrm{cond}}(S, \mathbb{Z}) = \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{Z})$, so we have to show $\mathrm{H}^{\geq 1}_{\mathrm{cond}}(S, \mathbb{Z}) = 0$.

Pick an e.d. hypercover $S_\bullet \to S$, and for each S^j choose finite hypercover $S^j_\bullet \to S^j$ such that $S_n = \varprojlim_j S^j_n$.
Then S^j is extremally disconnected, so that

$$0 \to \Gamma(S^j, \mathbb{Z}) \to \Gamma(S^j_0, \mathbb{Z}) \to \Gamma(S^j_1, \mathbb{Z}) \to \cdots$$

is exact. Taking filtered colimits shows exactness of

$$0 \to \Gamma(S, \mathbb{Z}) \to \Gamma(S_0, \mathbb{Z}) \to \Gamma(S_1, \mathbb{Z}) \to \cdots .$$

Thus $\mathrm{H}^{\geq 1}_{\mathrm{cond}}(S, \mathbb{Z}) = 0$.

The morphism

Consider morphism of topoi

$$\alpha : (\text{sheaves over } \mathbf{Comp}/S) \to (\text{sheaves on } S).$$

For an abelian sheaf $\mathcal{F}$ over $\mathbf{Comp}/S$, $\alpha_*(\mathcal{F})$ is the following abelian sheaf on S

$$U \mapsto \varprojlim_{U \supseteq V \text{ closed in } S} \mathcal{F}(V \hookrightarrow S).$$

α_* is left exact and $\Gamma_{\text{sheaf}}(S, \cdot) \circ \alpha_* = \Gamma_{\text{cond}}(S, \cdot)$.

We have to show $R\alpha_*\mathbb{Z} \cong \mathbb{Z}$ in $D(\text{abelian sheaves on } S)$, as then

$$\begin{aligned} \mathrm{H}^\bullet_{\text{cond}}(S, \mathbb{Z}) &= \mathrm{H}^\bullet(R\Gamma_{\text{cond}}(S, \mathbb{Z})) = \mathrm{H}^\bullet(R\Gamma_{\text{sheaf}}(S, \cdot) \circ R\alpha_*(\mathbb{Z})) \\ &\overset{*}{=} \mathrm{H}^\bullet(R\Gamma_{\text{sheaf}}(S, \mathbb{Z})) = \mathrm{H}^\bullet_{\text{sheaf}}(S, \mathbb{Z}). \end{aligned}$$

Towards $R\alpha_*\mathbb{Z}$

$R\alpha_*\mathbb{Z}$ is a complex of abelian sheaves on S.
$\mathrm{H}^0(R\alpha_*\mathbb{Z}) \cong \alpha_*\mathbb{Z}$ as abelian sheaves on S.

The global sections $\Gamma_{\mathsf{sheaf}}(S, \mathrm{H}^0(R\alpha_*\mathbb{Z})) \cong \Gamma_{\mathsf{cond}}(S, \mathbb{Z})$ induce a morphism of sheaves $\mathbb{Z} \to \mathrm{H}^0(R\alpha_*\mathbb{Z}))$.

This yields a morphism of complexes of abelian sheaves

$$\mathbb{Z} \text{ (concentrated in degree 0)} \to R\alpha_*\mathbb{Z}.$$

We prove this is an isomorphism on stalks. Fix $s \in S$.

Computing $(R\alpha_*\mathbb{Z})_s$

$$(R\alpha_*\mathbb{Z})_s = \varinjlim_{s\in U \text{ open}} R\Gamma(U, R\alpha_*\mathbb{Z}) = \varinjlim_{s\in V \text{ closed nbh}} R\Gamma_{\text{cond}}(V, \mathbb{Z}).$$

Pick a hypercover $S_\bullet \to S$ by profinite (extremally disconnected) sets.

For each closed V, $(S_n \times_S V)_{n\geq 0} \to V$ is a hypercover by profinite sets. Hence $R\Gamma_{\text{cond}}(V, \mathbb{Z})$ is isomorphic to

$$0 \to \Gamma(S_0 \times_S V, \mathbb{Z}) \to \Gamma(S_1 \times_S V, \mathbb{Z}) \to \Gamma(S_2 \times_S V, \mathbb{Z}) \to \cdots$$

Taking the filtered colimit over $V \ni s$ yields

$$\begin{aligned} & 0 \to \varinjlim_{V\ni s} \Gamma(S_0 \times_S V, \mathbb{Z}) \to \varinjlim_{V\ni s} \Gamma(S_1 \times_S V, \mathbb{Z}) \to \cdots \\ \cong \quad & 0 \to \Gamma(S_0 \times_S \{s\}, \mathbb{Z}) \to \Gamma(S_1 \times_S \{s\}, \mathbb{Z}) \to \cdots \\ \cong \quad & R\Gamma_{\text{cond}}(\{s\}, \mathbb{Z}) \cong \mathbb{Z}. \end{aligned}$$

Proof summary

Proving the claim (or rather, acyclicity) for profinite sets allows us to use profinite hypercovers, which can be restricted to closed subsets $V \subseteq S$.

Suffices to check $R\alpha_*\mathbb{Z} \cong \mathbb{Z}$.

Found a morphism $\mathbb{Z} \to R\alpha_*\mathbb{Z}$ and then checked isomorphism property on stalks (well, "checked"). □

The result

By $\mathbb{R}$, we denote the condensed abelian group sending $S \in \mathbf{Comp}$ to $C(S, \mathbb{R}) = \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{R})$ *with real topology*.

Theorem.

We have

$$\mathrm{H}^0_{\mathrm{cond}}(S, \mathbb{R}) = C(S, \mathbb{R}), \qquad \mathrm{H}^{\geq 1}_{\mathrm{cond}}(S, \mathbb{R}) = 0.$$

More precisely, if $S_\bullet \to S$ *is a profinite hypercover, the complex*

$$0 \to C(S, \mathbb{R}) \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \cdots$$

satisfies a quantified version of exactness:
For $f \in C(S_i, \mathbb{R})$ *with* $d(f) = 0$ *and* $\varepsilon > 0$, *can write* $f = d(g)$ *with*

$$\|g\|_\infty := \max_{s \in S} |g(s)| \leq (i + 2 + \varepsilon)\|f\|_\infty.$$

Remarks

Of course, this is not $\mathrm{H}^\bullet_{\text{sheaf}}(S, \mathbb{R}) = \mathrm{H}^\bullet(S, \mathbb{Z}) \otimes \mathbb{R}$ because we respect the topology on $\mathbb{R}$.

For the sheaf $\mathcal{F} = C(\cdot, \mathbb{R})$, we have $\mathrm{H}^{\geq 1}_{\text{sheaf}}(S, \mathcal{F}) = 0$ as $\mathcal{F}$ is *soft*. Thus $\mathrm{H}^\bullet_{\text{sheaf}}(S, \mathcal{F}) = \mathrm{H}^\bullet_{\text{cond}}(S, \mathcal{F})$.

For any morphism of compact spaces $S \xrightarrow{\varphi} S'$, the induced map $C(S', \mathbb{R}) \xrightarrow{\varphi^*} C(S, \mathbb{R})$ has norm at most 1:

$$\|\varphi^*(f)\|_\infty = \|f \circ \varphi\|_\infty \leq \|f\|_\infty \text{ for all } f \in C(S', \mathbb{R}).$$

Finite case

Let S and all S_i be finite.

Another finite hypercover of S is given by $S'_j := S$, all structure maps being the identity.

These two hypercovers are *homotopy-equivalent* as simplicial sets, inducing a homotopy equivalence of the complexes

$$\begin{array}{ccccccccc} 0 \to C(S,\mathbb{R}) & \longrightarrow & C(S_0,\mathbb{R}) & \longrightarrow & C(S_1,\mathbb{R}) & \longrightarrow & \cdots \\ \| & & \uparrow\downarrow & & \uparrow\downarrow & & \\ 0 \to C(S,\mathbb{R}) & \xrightarrow{\text{id}} & C(S,\mathbb{R}) & \xrightarrow{0} & C(S,\mathbb{R}) & \xrightarrow{\text{id}} & \cdots \end{array}$$

Finite case, continued

$$\begin{array}{ccccccccc} 0 \to C(S,\mathbb{R}) & \longrightarrow & C(S_0,\mathbb{R}) & \longrightarrow & C(S_1,\mathbb{R}) & \longrightarrow & \cdots \\ \| & & \uparrow\downarrow & & \uparrow\downarrow & & \\ 0 \to C(S,\mathbb{R}) & \xrightarrow{\text{id}} & C(S,\mathbb{R}) & \xrightarrow{0} & C(S,\mathbb{R}) & \xrightarrow{\text{id}} & \cdots \end{array}$$

In the lower complex, every f with $df = 0$ has preimage g with $\|g\|_\infty = \|f\|_\infty$.

The chain homotopy $h_i : C(S_i,\mathbb{R}) \to C(S_{i-1},\mathbb{R})$ is the alternating sum of $i+1$ pullback maps. Thus h_i has norm $\leq i+1$.

Combining these results, each $f \in C(S_i,\mathbb{R})$ with $df = 0$ has a preimage $g \in C(S_{i-1},\mathbb{R})$ with $\|g\|_\infty \leq (i+2)\|f\|_\infty$.

Profinite case

Let S and each S_i be profinite.

Write $S = \varprojlim_j S^j$ and $S_i = \varprojlim_j S_i^j$ with each S^j, S_i^j finite and $S_\bullet^j \to S^j$ a hypercover.

By the previous case,

$$0 \to C(S^j, \mathbb{R}) \to C(S_0^j, \mathbb{R}) \to C(S_1^j, \mathbb{R}) \to \cdots$$

is exact, and each cocycle f in degree $i \geq 0$ can be written as $f = dg$ with $\|g\|_\infty \leq (i+2)\|f\|_\infty$.

Passing to the filtered colimit, the same holds for

$$0 \to \varinjlim_j C(S^j, \mathbb{R}) \to \varinjlim_j C(S_0^j, \mathbb{R}) \to \varinjlim_j C(S_1^j, \mathbb{R}) \to \cdots$$

Completion

We have a morphism of complexes

$$\begin{array}{ccccccccc} 0 \to & \varinjlim_i C(S^j, \mathbb{R}) & \to & \varinjlim_j C(S_0^j, \mathbb{R}) & \to & \varinjlim_j C(S_1^j, \mathbb{R}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & C(S, \mathbb{R}) & \longrightarrow & C(S_0, \mathbb{R}) & \longrightarrow & C(S_1, \mathbb{R}) & \longrightarrow \cdots \end{array}$$

such that each $\varinjlim_i C(S_i^j, \mathbb{R}) \to C(S_i, \mathbb{R})$ is an isometric and dense embedding, i.e. completion map of normed vector spaces.

Let now $f \in C(S_i, \mathbb{R})$ satisfy $df = 0$. Pick a first approximation $f^{(1)} \in \varinjlim_i C(S_i^j, \mathbb{R})$ with $\|f^{(1)} - f\|_\infty$ "sufficiently small".

Approximation

$f \in C(S_i, \mathbb{R})$ satisfies $df = 0$. Have an approximation $f^{(1)} \in \varinjlim_i C(S_i^j, \mathbb{R})$ with $\|f^{(1)} - f\|_\infty$ "small".

Then $\|df^{(1)}\|_\infty = \|d(f^{(1)} - f)\|_\infty \leq (i+2)\|f^{(1)} - f\|_\infty$ is "small", so we find (similar to previous proof) $g^{(1)} \in \varinjlim_j C(S_{i-1}^j, \mathbb{R})$ with $\|g^{(1)}\|_\infty \leq (i+2)\|f^{(1)}\|_\infty$ and $\|dg^{(1)} - f^{(1)}\|_\infty$ "small".

In particular, we can make $\|f - dg^{(1)}\|_\infty$ arbitrarily small.

Pick an approximation $f^{(2)}$ for $f - dg^{(1)}$ and repeat.
Now $\|g^{(n)}\|_\infty \to 0$ rapidly, so that $g := \sum_n g^{(n)}$ exists in $C(S_{i-1}, \mathbb{R})$.
Then $dg = f$ and

$$\|g\|_\infty \leq \sum_n \|g^{(n)}\|_\infty \leq (i+2)\sum_n \|f^{(n)}\|_\infty \leq (i+2+\varepsilon)\|f\|_\infty$$

if we choose our approximations sufficiently good.

Local approximations

Let S be a compact space and $S_\bullet$ a profinite hypercover.
Let $f \in C(S_i, \mathbb{R})$ satisfy $df = 0$.

Pick $s \in S$, we first find an approximation "near" s:
The hypercover $S_\bullet \underset{S}{\times} \{s\} \to \{s\}$ is handled by the above proof, so we find $g_s \in C(S_{i-1} \times_S \{s\}, \mathbb{R})$ with

$$dg_s = f|_{S_{i-1} \times_S \{s\}}, \qquad \|g_s\|_\infty \leq (i + 2 + \varepsilon) \|f|_{S_i \times_S \{s\}}\|_\infty.$$

Extend g_s to a continuous function $\tilde{g}_s : S_{i-1} \to \mathbb{R}$.
Then $(d\tilde{g}_s - f)(S_i \times_S \{s\}) = 0$, so there exists an open neighbourhood $U_s \ni s$ with $\|(d\tilde{g}_s - f)|_{S_i \times_S U_s}\|_\infty$ "small".

By compactness, finitely many such neighbourhoods cover S:

We can cover $S = \bigcup_{j=1}^n U_j$ with functions $g_j \in C(S_{i-1}, \mathbb{R})$ such that $\|(dg_j - f)|_{S_i \times_S U_j}\|_\infty$ is "small" and $\|g_j\|_\infty \leq (i + 2 + \varepsilon)\|f\|_\infty$.

Glueing approximations

We can cover $S = \bigcup_{j=1}^{n} U_j$ with functions $g_j \in C(S_{i-1}, \mathbb{R})$ such that $\|(dg_j - f)|_{S_i \times_S U_j}\|_\infty$ is "small" and $\|g_j\|_\infty \leq (i + 2 + \varepsilon)\|f\|_\infty$.

Pick a partition of unity ρ for this cover, i.e. compactly supported $\rho_j \in C(U_j, [0,1])$ with $1 = \sum_j \rho_j$. Pullback to a partition of unity on S_i and S_{i-1}.

$$g^{(1)} := \sum_j g_j \rho_j.$$

$$\implies \|g^{(1)}\|_\infty \leq (i + 2 + \varepsilon)\|f\|_\infty,$$

$$\|f - dg^{(1)}\|_\infty = \|\sum_j \rho_j (f - dg_j)\|_\infty \leq \max_j \|(dg_j - f)|_{S_i \times_S U_j}\|_\infty.$$

Repeat for $f^{(2)} = f - dg^{(1)}$. Similar arguments as before show that $g := \sum_m g^{(m)}$ satisfies $dg = f$ and

$$\|g\|_\infty \leq (i + 2 + \text{constant} \cdot \varepsilon)\|f\|_\infty.$$

□

Summary and References

Singular cohomology is bad for our purposes.

Čech=sheaf cohomology is good for $\mathbb{Z}$ and $C(\cdot, \mathbb{R})$.

There is a quantitative exactness result for $\mathbb{R}$.

The precise definitions of "small" are in Scholze's lecture notes.

A gentle (but lengthy) introduction to simplicial sets, hypercovers and much more is
`https://math.stanford.edu/~conrad/papers/hypercover.pdf`.

Thank you for listening!