Universität Regensburg
Fakultät für Mathematik

WS 19–20

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Oberseminar Motivic Sheaves:
Microsupport and Characteristic Cycles
Monday 10 – 12, Room M 102

Talk 1 – Introduction. [Denis-Charles Cisinski, 14-10-2019]
We will discuss the precise goal of the seminar for the whole year, and distribute talks for the Wintersemester. Basically, for the Wintersemester, we shall follow parts of second half of the book of Kashiwara and Schapira [KS94], focusing on the geometrical and functorial apsects which are useful to define and control the microsupport (aka singular support) as well as to define characteristic cycles. For the Sommersemester, we will explain how to do with algebraic varieties over a field an analogue of what Kashiwara and Schapira did with real/complex analytic manifolds.

Talk 2 – Constructible sheaves and the γ-topology. [Kévin François, 21-10-2019]
In this basic talk, we will review the notion of constructible sheaves on a locally compact space as well as the γ-topology on a real finite dimensional vector space. Namely, in the first part of the talk, we will recall [KS94, Def. 3.4.1], state (and possibly give an idea of the proofs of) basic results on duality and Künneth formulas in this context [KS94, Prop. 3.4.3, 3.4.4, 3.4.6]. Then, in a second part, we will introduce the γ-topology [KS94, Def. 3.5.1] and discuss its nice cohomological properties [KS94, 3.5.3 and 3.5.4]. This will play a central role later in our understanding of microlocalization.

Talk 3 – Fourier-Sato transformation I. [Marco Volpe, 28-10-2019]
We will first study the general theory of functors defined through integration with a kernel, following [KS94, § 3.6]. We will then introduce *conic sheaves* [KS94, Def. 3.7.1] and observe that conic sheaves may be determined by a descent property with respect to scalar multiplication: recall [KS94, Def. 2.7.4, Prop. 2.7.5, Cor. 2.7.7] and then prove [KS94, Prop. 3.7.2, Cor. 3.7.3]. After that, we will see basic functoriality properties of conic sheaves: [KS94, Prop. 3.7.4, 3.7.5].

Talk 4 – Fourier-Sato transformation II. [Masoud Zargar, 04-11-2019]
We will introduce the two candidates for the Fourier-Sato transformation and prove they are isomorphic [KS94, Thm. 3.7.7]. After introducing the proper definition of the Fourier-Sato formula, we will prove the analogue of Plancherel theorem (aka fully faithfulness) [KS94, Thm. 3.7.9] as well as the analogue of the inversion formula [KS94, Prop. 3.7.2], after proving the preparatory [KS94, Lemma 3.7.10]. We will finish with further functoriality properties [KS94, Prop. 3.7.13, 3.7.14, 3.7.15].

Talk 5 – Deformation to the normal cone. [Uli Bunke, 11-11-2019]
We will introduce the appropriate version of deformation to the normal cone for our context [KS94, 4.1.1] and explain its main geometrical properties [KS94, 4.1.2–4.1.4]. It makes sense to spend some time on Prop. 4.1.4 and its proof because it will plays a central role later. We will then introduce the notions of clean map and of transversality [KS94, 4.1.5, 4.1.6].

Talk 6 – Specialization. [Benedikt Preis, 18-11-2019]
We may start by a short reminder on [KS94, Prop. 3.3.9 and Rem. 3.3.10]. Then, recalling the notations of [KS94, (4.1.5)], we may introduce the specialization functor [KS94, 4.2.2] and observe that it has an easy alternative presentation [KS94, Lemma 4.2.1]. We will then spend some time on the proof of [KS94, Thm. 4.2.3] which computes the fibers of specialization and, very crucially, its restriction on the zero-section. We will then introduce various natural comparison maps provided by [KS94, 4.2.4–4.2.7].

Talk 7 – Microlocalization. [25-11-2019]
We will define the microlocalization functor [KS94, Def. 4.3.1] and state its basic properties [KS94, 4.3.2–4.3.8]. We will then introduce the *μhom* functor [KS94, 4.4.1] and see how it relates with classical constructions [KS94, 4.4.1–4.4.2]. We will then see two important formulas: one which relates *μhom* with the microlocalization functor [KS94, Prop. 4.4.3], and another one which computes the stalks of *μhom* in terms of the γ-topology [KS94, Prop. 4.4.4].

Talk 8 – Microsupport I. [2.12.2019]
In this talk, we will introduce the microsupport. We will first prove [KS94, Prop. 5.1.1] and then give [KS94, Def. 5.1.2] and state [KS94, Prop. 5.1.3]. We will then prove two important properties related to the γ-topology: [KS94, Prop. 5.2.1 and 5.2.3].

Talk 9 – Microsupport II. [9-12-2019]
In this talk, we will give examples of microsupports [KS94, 5.3] and then prove basic functoriality properties: [KS94, Rem. 5.1.4, Prop. 5.4.1–5.4.8, Cor. 5.4.9–5.4.11].

Talk 10 – Microsupport III. [16-12-2019]
We will introduce non-characteristic morphism of manifolds relatively to a conic subset of the cotangent bundle on its codomain [KS94, Def. 5.4.12], and prove that it corresponds to a notion of transversality which is controled by the singular support [KS94, Prop. 5.4.13, 5.4.14, 5.4.17]. We will then study the microsupport of conic sheaves, following closely [KS94, § 5.5]. In particular, we will observe the compatibility of the singular support with the Fourier-Sato tranformation [KS94, Thm. 5.5.5].

Talk 11 – Microsupport IV. [13-01-2020]
Here we will read [KS94, § 6.1] but interpret the constructions in terms of stable ∞-categories. In particular, we will prove [KS94, 6.1.2] as a global statement: *μhom* computes the maps in the sheaf of stable ∞-categories given by sheafifying the presheaf $\Omega \mapsto \mathrm{D}^b(\mathrm{X};\Omega)$ provided by [KS94, Def. 6.1.1]. In particular, the microsupport may be recovered as the support of the microlocalization [KS94, Cor. 6.1.3]. We will then study the microlocal cut-off lemma [KS94, 6.1.4] and its dual [KS94, 6.1.6].

Talk 12 – Normal cones in cotangent bundles. [20-01-2020]
We will start by introducing basic operations on conic subsets [KS94, Def. 6.2.3]. We will explain (but not prove) [KS94, Lem. 6.2.1] and prove various basic properties of these operations [KS94, Prop. 6.2.4, Rem. 6.2.5, 6.2.6]. We will then proceed to the compatibility with operations of sheaves through the microsupport: [KS94, Def. 6.2.7, Rem. 6.2.8, Theorem 6.3.1, Prop. 6.3.2]. We will finish with [KS94, Thm. 6.4.1], relating the various versions of the microsupport with microlocalization and specialization.

Talk 13 – Constructible sheaves. [George Raptis, 27-01-2020]
We will follow [KS94, § 8.4], except that, we will skip Theorem 8.4.2, and, in Definition 8.5.3, we will take assertion (i) of Theorem 8.4.2 a definition of weakly constructible. In particular, we will get new formulas relating specialization, microlocalization and *μhom* with duality. If time permits, we might briefly describe the complex case [KS94, § 8.5]. Constuctible sheaves are important because they are the 'analytic motives', so that we might consider them as a rather good model of what might hapen in an algebraic context.

Talk 14 – Vanishing cycle functor. [3-02-2020]
We will introduce the nearby cycle functor [KS94, Def. 8.6.1] and the vanishing cycle functor [KS94, Def. 8.6.2]. We will explain in detail the interplay between those and the specialization functor [KS94, Prop. 8.6.3]. Finally, we will see how to use the vanishing cycle functor to determine the microsupport [KS94, Prop. 8.6.4].

References

[Bea08] A. Borel and et al. *Intersection cohomology*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition.

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.