Cohomological Methods in Intersection Theory

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Abstract These notes are an account of a series of lectures I gave at the LMS-CMI Research School 'Homotopy Theory and Arithmetic Geometry: Motivic and Diophantine Aspects', in July 2018, at the Imperial College London. The goal of these notes is to see how motives may be used to enhance cohomological methods, giving natural ways to prove independence of ℓ results and constructions of characteristic classes (as 0-cycles). This leads to the Grothendieck-Lefschetz formula, of which we give a new motivic proof. There are also a few additions to what have been told in the lectures:

- A proof of Grothendieck-Verdier duality of étale motives on schemes of finite type over a regular quasi-excellent scheme (which slightly improves the level of generality in the existing literature).
- A proof that $\mathbf{Q}$-linear motivic sheaves are virtually integral (Theorem 2.2.12).
- A proof of the motivic generic base change formula.

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Introduction

Let p be a prime number and $q = p^r$ a power of p. Let X_0 be a smooth and projective algebraic variety over $\mathbf{F}_q$. It comes equipped with the geometric Frobenius map $\phi_r : X \to X$, where $X = X_0 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_p$, so that the locus of fixed points of F

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corresponds to the set of rational points of X_0 (various Frobenius morphisms and their actions are discussed in detail in Remark 3.2.24 below). We may take the graph of Frobenius $\Gamma_{\phi_r} \subset X \times X$, intersect with the diagonal, then interpret the intersection number cohomologically with the formula of Lefschetz through ℓ-adic cohomology, with $\ell \neq p$.

For each $Z \subseteq X$ we can attach a cycle $[Z] \in H^*(X, \mathbf{Q}_\ell)$ and do intersection theory (interpreting geometrically the algebraic operations on cycle classes). For instance, if $Z' \subseteq X$ is another cycle which is transversal to Z, we have $[Z] \cdot [Z'] = [Z \cap Z']$. Together with Poincaré duality, this implies that the number of rational points of X_0 may be computed cohomologically:

$$\#X(\mathbf{F}_q) = \sum_i (-1)^i \operatorname{Tr}\big(\phi_r^* : H^i(X, \mathbf{Q}_\ell) \to H^i(X, \mathbf{Q}_\ell)\big) .$$

The construction of ℓ-adic cohomology by Grothendieck was aimed precisely at proving this kind of formulas, with the goal of proving Weil's conjectures on the ζ-functions of smooth and projective varieties over finite fields, which was finally achieved by Deligne [Del73, Del80].

Here are two natural problems we would like to discuss:

- Extend this to non-smooth or non-proper schemes and to cohomology with possibly non-constant coefficients: this is what the Grothendieck-Lefschetz formula is about.
- Address the problem of independance on ℓ (when we compute traces of endomorphisms with a less obvious geometric meaning): this is what motives are made for.

In this series of lectures, I will explain what is a motive and explain how to prove a motivic Grothendieck-Lefschetz formula. To be more precise, we shall work with *h-motives* over a scheme X, which are one of the many descriptions of étale motives. These are the objects of the triangulated category $DM_h(X)$ constructed and studied in details in [CD16], which is a natural modification (the non-effective version) of an earlier construction of Voevodsky [Voe96], following the lead of Morel and Voevodsky into the realm of $\mathbf{P}^1$-stable $\mathbf{A}^1$-homotopy theory of schemes. Although we will not mention them in these notes, we should mention that there are other equivalent constructions of étale motives which are discussed in [CD16] and [Ayo14] (not to speak of the many models with $\mathbf{Q}$-coefficients discussed in [CD19]), and more importantly, that there are also other flavours of motives [VSF00, Kel17, CD15], which are closer to geometry (and further from topology), for which one can still prove Lefschetz-Verdier formulas; see [JY18]. As we will see later, étale motives with torsion coefficients may be identified with classical étale sheaves. In particular, when restricted to the case of torsion coefficients, all the results discussed in these notes on trace formulas go back to Grothendieck [Gro77]. The case of rational coefficients has also been studied previously to some extend by Olsson [Ols16, Ols15]. We will see here how these fit together, as statements about étale motives with arbitrary (e.g. integral) coefficients. Finally, we will recall the Lefschetz-Verdier trace formula and explain how to deduce from it the motivic Grothendieck-Lefschetz formula, using Bondarko's theory of weights and Olsson's computations of local terms of the motivic Lefschetz-Verdier trace.

1 Étale motives

1.1 The h-topology

Definition 1.1.1 A morphism of schemes $f : X \to Y$ is a *universal topological isomorphism* (*epimorphism* resp.) if for any map of schemes $Y' \to Y$, the pullback $X' = Y' \times_Y X \to Y'$ is a homeomorphism (a topological epimorphism resp., which means that it is surjective and exhibits Y' as a topological quotient).

Example 1.1.2 Surjective proper maps as well as faithfully flat maps all are universal epimorphisms.

Proposition 1.1.3 *A morphism of schemes $f : X \to Y$ is a universal homeomorphism if and only if it is surjective radicial and integral. Namely, f is integral and, for any algebraically closed field K, induces a bijection $X(K) \cong Y(K)$.*

Example 1.1.4 The map $X_{red} \to X$ is a universal homeomorphism.

Example 1.1.5 Let K'/K be a purely inseparable extension of fields. If X is a normal scheme with field of functions K, and if X' is the normalization of X in K', then the induced map $X' \to X$ is a universal homeomorphism.

Following Voevodsky [Voe96], we can define the *h-topology* as the Grothendieck topology on noetherian schemes associated to the pre-topology whose coverings are finite families $\{X_i \to X\}_{i \in I}$ such that the induced map $\coprod_i X_i \to X$ is a universal epimorphism.[1] Beware that the h-topology is not subcanonical: any universal homeomorphism becomes invertible locally for the h-topology.

Using Raynaud-Gruson's flatification theorem, one shows the following [Ryd10]:

Theorem 1.1.6 *(Voevodsky, Rydh): Let $X_i \to X$ be an h-covering. Then there exists an open Zariski cover $X = \cup_j X_j$ and for each j a blow-up $U'_j = Bl_{Z_j} U_j$ for some closed subset $Z_j \subseteq U_j$, a finite faithfully flat $U''_j \to U'_j$ and a Zariski covering $\{V_{j,\alpha}\}_\alpha$ of U''_j such that we have a dotted arrow making the following diagram commutative.*

$$\begin{array}{ccccccc}
\coprod_{j,\alpha} V_{j,\alpha} & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \dashrightarrow & \coprod_i X_i \\
\downarrow & & & & & & \downarrow \\
\coprod_j U''_j & \longrightarrow & \coprod_j U'_j & \longrightarrow & \coprod_j U_j & \longrightarrow & X
\end{array}$$

[1] As shown by D. Rydh [Ryd10], this topology can be extended to all schemes, at the price of adding compatiblities with the constructible topology.

This means that the property of descent with respect to the h-topology is exactly the property of descent for the the Zariski topology, together with proper descent.

Remark 1.1.7 Although Grothendieck topologies where not invented yet, a significant amount of the results of SGA 1 [Gro03] are about h-descent of étale sheaves (and this is one of the reasons why the very notion of descent was introduced in SGA 1). This goes on in SGA 4 [AGV73] where the fact that proper surjective maps and étale surjective maps are morphism of universal cohomological descent is discussed at length. However, it is only in Voevodsky's thesis [Voe96] that the h-topology is defined and studied properly, with the clear goal to use it in the definition of a triangulated category of étale motives.

1.2 Construction of motives, after Voevodsky

1.2.1 Let Λ be a commutative ring. Let $Sh_h(X,\Lambda)$ denote the category of sheaves of Λ-modules on the category of separated schemes of finite type over X with respect to the h-topology. We have Yoneda functor

$$Y \mapsto \Lambda(Y)\,,$$

where $\Lambda(Y)$ is the h-sheaf associated to the presheaf $\Lambda[\mathrm{Hom}_X(-,Y)]$ (the free Λ module generated by $\mathrm{Hom}_X(-,Y)$).

Let us consider the derived category $D(Sh_h(X,\Lambda))$, i.e. the localization of complexes of sheaves by the quasi-isomorphisms. Here we will speak the language of ∞-categories right away.[2] In particular, the word 'localization' has to be interpreted higher categorically (if we take as models simplicial categories, this is also known as the Dwyer-Kan localization). That means that $D(Sh_h(X,\Lambda))$ is in fact a stable ∞-category with small limits and colimits (as is any localization of a stable model category). Moreover, the constant sheaf functor turns it into an ∞-category enriched in the monoidal stable ∞-category $D(\Lambda)$ of complexes of Λ-modules (i.e. the localization of the category of chain complexes of Λ-modules by the class of quasi-isomorphisms). In particular, for any objects $\mathcal{F}$ and $\mathcal{G}$ of $D(Sh_h(X,\Lambda))$, morphisms from $\mathcal{F}$ to $\mathcal{G}$ form an object $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ of $D(\Lambda)$. The appropriate version of the Yoneda Lemma thus reads:

$$\mathrm{Hom}(\Lambda(Y),\mathcal{F}) \cong \mathcal{F}(Y)$$

for any separated X-scheme of finite type Y. In particular, $H^i(Y,\mathcal{F}) = H^i(\mathcal{F}(Y))$ is what the old fashioned literature would call the i-th hypercohomology group of Y with coefficients in $\mathcal{F}$.

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[2] We refer to [Lur09, Lur17] in general. However, most of the literature on motives is written using the theory of Quillen model structures. The precise way to translate this language to the one of ∞-categories is discussed in Chapter 7 of [Cis19].

1.2.2 A sheaf $\mathcal{F}$ is called $\mathbf{A}^1$-*local* if $\mathcal{F}(Y) \to \mathcal{F}(Y \times \mathbf{A}^1)$ is an equivalence for all Y. A map $f : M \to N$ is an $\mathbf{A}^1$-*equivalence* if for every $\mathbf{A}^1$-local $\mathcal{F}$ the map

$$f^* : \mathrm{Hom}(N, \mathcal{F}) \to \mathrm{Hom}(M, \mathcal{F})$$

is an equivalence.

Define

$$\underline{DM}_h^{eff}(X, \Lambda)$$

to be the localization of $D(Sh_h(X, \Lambda))$ with respect to $\mathbf{A}^1$-equivalences. We have a localization functor $D(Sh_h(X, \Lambda)) \to \underline{DM}_h^{eff}(X, \Lambda)$ with fully faithfull right adjoint whose essential image consists of the $\mathbf{A}^1$-local objects. An explicit description of the right adjoint is by taking the total complex of the bicomplex

$$C_*(\mathcal{F})(Y) = \cdots \to \mathcal{F}(Y \times \Delta^n_{\mathbf{A}^1}) \to \cdots \to \mathcal{F}(Y \times \Delta^1_{\mathbf{A}^1}) \to \mathcal{F}(Y),$$

where $\Delta^n_{\mathbf{A}^1} = Spec(k[x_0, \dots, x_n]/(x_0 + \cdots + x_n = 1))$. The ∞-category $\underline{DM}_h^{eff}(X, \Lambda)$ comes equipped with a canonical functor

$$\gamma_X : Sch/X \times D(\Lambda) \to \underline{DM}_h^{eff}(X, \Lambda)$$

defined by $\gamma_X(Y, K) = \Lambda(Y) \otimes_\Lambda K$. Furthermore, it is a presentable ∞-category (as a left Bousfield localization of a presentable ∞-category, namely $D(Sh_h(X, \Lambda))$), and thus has small colimits and small limits. For a cocomplete ∞-category C, the category of colimit preserving functors $\underline{DM}_h^{eff}(X, \Lambda) \to C$ is equivalent to the category of functors $F : Sch/X \times D(\Lambda) \to C$ with the following two properties:

- For each X-scheme Y, the functor $F(Y, -) : D(\Lambda) \to C$ commutes with colimits.
- For each complex of Λ-modules K, we have:
  a) the first projection induces an equivalence $F(Y \times \mathbf{A}^1, K) \cong F(Y, K)$ for any X-scheme Y;
  b) for any h-hypercovering U of Y, the induced map $\mathrm{colim}_{\Delta^{op}} F(U, K) \to F(Y, K)$ is invertible.

The functor $\underline{DM}_h^{eff}(X, \Lambda) \to C$ associated to such an F is constructed as the left Kan extension of F along γ_X.

There is still an issue. Indeed, let $\infty \in \mathbf{P}^1$ and let us form the following cofiber sequence:

$$\Lambda(X) \xrightarrow{\infty} \Lambda(\mathbf{P}^1) \to \Lambda(1)[2]$$

In order to express Poincaré duality (or, more generally, Verdier duality), we need the cofiber $\Lambda(1)[2]$ above to be $\otimes$-invertible. But it is not so.

Definition 1.2.3 An object $A \in C$ is $\otimes$-*invertible* if the functor $A \otimes - : C \to C$ is an equivalence of ∞-categories.

We want to invert a non-invertible object. Let us think about the case of a ring.

$$R[f^{-1}] = \mathrm{colim}(R \xrightarrow{f} R \xrightarrow{f} \cdots)$$

(The colimit is taken within R-modules.) For ∞-categories, we define $C[A^{-1}]$ with a similar colimit formula. Note however that the colimit needs to be taken in the category of presentable ∞-categories (in which the maps are the colimit preserving functors). We get an explicit description of this colimit as follows. For C presentable, $C[A^{-1}]$ can be described as the limit of the diagram

$$\cdots \xrightarrow{Hom(A,-)} C \xrightarrow{Hom(A,-)} C \xrightarrow{Hom(A,-)} C$$

in the ∞-category of ∞-categories (here, $Hom(A,-)$ is the right adjoint of the functor $A \otimes -$). Therefore, an object in $C[A^{-1}]$ is typically a sequence $(M_n, \sigma_n)_{n\geq 0}$ with M_n objects of C and $\sigma_n : M_n \xrightarrow{\sim} Hom(A, M_{n+1})$ equivalences in C. Note that, in the case where A is the circle in the ∞-category of pointed homotopy types, we get exactly the definition of an Ω-spectrum from topology. There is a canonical functor

$$\Sigma^\infty : C \to C[A^{-1}]$$

which is left adjoint to the functor

$$\Omega^\infty : C[A^{-1}] \to C$$

defined as $\Omega^\infty(M) = M_0$ where $M = (M_n, \sigma_n)_{n\geq 0}$ is a sequence as above.

There is still the issue of having a natural symmetric monoidal structure on $C[A^{-1}]$, which is not automatic. However, if the cyclic permutation acts as the identity on $A^{\otimes 3}$ (by permuting the factors) in the homotopy category of C, then there is a unique symmetric monoidal structure on $C[A^{-1}]$ such that the canonical functor $\Sigma^\infty : C \to C[A^{-1}]$ is symmetric monoidal (all these issues are very well explained in Robalo's [Rob15]). Fortunately for us, Voevodsky proved that this extra property holds for $C = \underline{DM}_h^{\mathit{eff}}(X,\Lambda)$ and $A = \Lambda(1)$.

Definition 1.2.4 The big category of h-motives is defined as:

$$\underline{DM}_h(X,\Lambda) = \underline{DM}_h^{\mathit{eff}}(X,\Lambda)[\Lambda(1)^{-1}].$$

Remark 1.2.5 However, what is important here is the universal property of the stable ∞-category $\underline{DM}_h(X,\Lambda)$; given a cocomplete ∞-category C, together with an equivalence of categories $T : C \to C$ each colimit preserving functor $\varphi : \underline{DM}_h^{\mathit{eff}}(X,\Lambda) \to C$ equipped with an invertible natural transformation $\varphi(M \otimes \Lambda(1)[2]) \cong T(\varphi(M))$ is the composition of a unique colimit preserving functor $\Phi : \underline{DM}_h(X,\Lambda) \to C$ equipped with an invertible natural transformation $\Phi(M \otimes \Sigma^\infty \Lambda(1)[2]) \cong T(\Phi(M))$.

Remark 1.2.6 We are very far from having locally constant sheaves here! In classical settings, the Tate object $\Lambda(1)$ is locally constant (more generally, for a smooth and proper map $f : X \to Y$ we expect each cohomology sheaf $R^i f_*(\Lambda)$ to be locally constant). However the special case of the projective line shows that we cannot have such a property motivically: Over field k, the cohomology with coefficients in $\mathbf{Q}$ vanishes in degree > 0, while, with coefficients in $\mathbf{Q}(1)$, it is equal to $k^\times \otimes \mathbf{Q}$ in degree 1. Therefore we should ask what is the replacement of locally constant sheaves. This will be dealt with later, when we will explain what are constructible motives.

Definition 1.2.7 We have an adjunction

$$\Sigma^\infty : \underline{DM}_h^{eff}(X, \Lambda) \rightleftarrows \underline{DM}_h(X, \Lambda) : \Omega^\infty$$

and we define $M(Y) = \Sigma^\infty \Lambda(Y)$. This is the *motive* of Y over X, with coefficents in Λ.

As we want eventually to do intersection theory, we need Chern classes within motives. Here is how they appear. Consider the morphisms of h-sheaves of groups $\mathbf{Z}(\mathbf{A}^1 - \{0\}) \to \mathbf{G}_m$ on the category Sch/X corresponding to the identity $\mathbf{A}^1 - \{0\} = \mathbf{G}_m$, seen as a map of sheaves of sets. From the pushout diagram

$$\begin{array}{ccc} \mathbf{A}^1 - \{0\} & \longrightarrow & \mathbf{A}^1 \\ \downarrow & & \downarrow \\ \mathbf{A}^1 & \longrightarrow & \mathbf{P}^1 \end{array}$$

and from the identification $\mathbf{Z} \cong \mathbf{Z}(\mathbf{A}^1)$, we get a (split) cofiber sequence

$$\mathbf{Z} \to \mathbf{Z}(\mathbf{A}^1 - \{0\}) \to \mathbf{Z}(1)[1]$$

Since the map $\mathbf{Z}(\mathbf{A}^1 - \{0\}) \to \mathbf{G}_m$ takes $\mathbf{Z}$ to 0, it induces a canonical map $\mathbf{Z}(1)[1] \to \mathbf{G}_m$.

Theorem 1.2.8 (Voevodsky) *The map* $\mathbf{Z}(1)[1] \to \mathbf{G}_m$ *is an equivalence in the effective category* $\underline{DM}_h^{eff}(X, \mathbf{Z})$.

As a result, we get canonical maps:

- h-hypersheafification:

$$Pic(X) = H^1_{Zar}(X, \mathbf{G}_m) \to H^0\mathrm{Hom}_{D(Sh_h(X,\Lambda))}(\mathbf{Z}, \mathbf{G}_m[1]) \,;$$

- $\mathbf{A}^1$-localization:

$$H^0\mathrm{Hom}_{D(Sh_h(X,\Lambda))}(\mathbf{Z}, \mathbf{G}_m[1]) \to H^0\mathrm{Hom}_{\underline{DM}_h^{eff}}(\mathbf{Z}, \mathbf{G}_m[1]) \,;$$

- $\mathbf{P}^1$-stabilization:

$$H^0\mathrm{Hom}_{\underline{DM}_h^{eff}(X,\mathbf{Z})}(\mathbf{Z}, \mathbf{G}_m[1]) \to H^0\mathrm{Hom}_{\underline{DM}_h(X,\mathbf{Z})}(\mathbf{Z}, \mathbf{Z}(1)[2]) \,.$$

By composition this gives us the first motivic Chern classes of line bundles.

$$c_1 : Pic(X) \to H^2_M(X, \mathbf{Z}(1)) = H^0\mathrm{Hom}_{\underline{DM}_h(X,\mathbf{Z})}(\mathbf{Z}, \mathbf{Z}(1)[2])$$

1.3 Functoriality

1.3.1 Recall that we have an assignment

$$X \mapsto \underline{DM}_h(X, \Lambda).$$

There is a unique symmetric monoidal structure on $\underline{DM}_h(X, \Lambda)$ such that the functor $M : Sch_{/X} \to \underline{DM}_h(X, \Lambda)$ is monoidal. It has the following properties (we write $\Lambda = M(X) \cong \Sigma^\infty(\Lambda)$ and $\Lambda(1) = \Sigma^\infty(\Lambda(1))$):

- $A(1) \cong A \otimes \Lambda(1)$; all functors of interest always commute with the functor $A \mapsto A(1)$.
- $M(Y \times \mathbf{P}^1) \cong M(Y)[2] \oplus M(Y)$.
- $A(n) = A \otimes \Lambda(n)$ is well defined for all $n \in \mathbf{Z}$ (with $\Lambda(n)$ the dual of $\Lambda(-n)$ for $n < 0$ and $\Lambda(0) = \Lambda$, $\Lambda(n+1) \cong \Lambda(n)(1)$ for $n \geq 0$).
- There is an internal Hom functor Hom.

For a morphism $f : X \to Y$ we have $f^* : \underline{DM}_h(Y, \Lambda) \to \underline{DM}_h(X, \Lambda)$ which preserves colimits and thus has right adjoint $f_* : \underline{DM}_h(X, \Lambda) \to \underline{DM}_h(Y, \Lambda)$. No property of f is required for that. We construct first the functor

$$f^* : \underline{DM}_h^{eff}(Y, \Lambda) \to \underline{DM}_h^{eff}(X, \Lambda)$$

as the unique colimit preserving functor which fits in the commutative diagram

$$\begin{array}{ccc} Sch/Y \times D(\Lambda) & \xrightarrow{f^* \times 1_{D(\Lambda)}} & Sch/X \times D(\Lambda) \\ \downarrow & & \downarrow \\ \underline{DM}_h^{eff}(Y, \Lambda) & \xrightarrow{f^*} & \underline{DM}_h^{eff}(X, \Lambda) \end{array}$$

(in which the vertical functors are the canonical ones $(U, C) \mapsto M(U) \otimes_\Lambda C$), and observe that it has a natural structure of symmetric monoidal functor. There is thus a unique symmetric monoidal pull-back functor f^* defined on $\underline{DM}_h$ so that the following squares commutes.

$$\begin{array}{ccc} \underline{DM}_h^{eff}(Y, \Lambda) & \xrightarrow{f^*} & \underline{DM}_h^{eff}(X, \Lambda) \\ \downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\ \underline{DM}_h(Y, \Lambda) & \xrightarrow{f^*} & \underline{DM}_h(X, \Lambda) \end{array}$$

If moreover f is separated and of finite type then the pull-back functor f^* has a left adjoint functor $f_\sharp : \underline{DM}_h(X, \Lambda) \to \underline{DM}_h(Y, \Lambda)$ which preserves colimits, and is essentially determined by the property that $f_\sharp M(U) = M(U)$ for any separated X-scheme of finite type U via universal properties as above. For example $f_\sharp(\Lambda) = M(X)$. We have a projection formula (proved by observing that the formula holds in the category of schemes and then extending by colimits)

$$f_\sharp(A \otimes f^*(B)) \xrightarrow{\simeq} f_\sharp A \otimes B.$$

Exercise 1.3.2 Show that, for any Cartesian square of noetherian schemes

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

and for any M in $\underline{DM}_h(X, \Lambda)$, if v is separated of finite type, then the canonical map

$$v^* f_*(M) \to f'_* u^*(M)$$

is invertible.

The base change formula above is too much: we want this to hold only for f proper of v smooth, because, otherwise, we will not have any good notion of support of a motive. This is why we have to restrict ourselves to a subcatefgory of $\underline{DM}_h(X, \Lambda)$, on which the support will be well defined.

Definition 1.3.3 Let $DM_h(X, \Lambda)$ be the smallest full subcategory of $\underline{DM}_h(X, \Lambda)$ closed under small colimits, containing objects of the form $M(U)(n)[i]$ for $U \to X$ smooth and $i, n \in \mathbf{Z}$.

Remark 1.3.4 The ∞-category $DM_h(X, \Lambda)$ is stable and presentable, essentially by construction. It is also stable under the operator $M \mapsto M(n)$ for all $n \in \mathbf{Z}$.

Theorem 1.3.5 (Localization Property) *Take $i : Z \to X$ to be a closed emdebbing with open complement $j : U \to X$ and let $M \in DM_h(X, \Lambda)$. Then we have a canonical cofiber sequence (in which the maps are the co-unit and unit of appropriate adjunctions):*

$$j_\sharp j^* M \to M \to i_* i^* M$$

Idea of the proof: the functors $j_\sharp$, j^*, i_* and i^* commute with colimits. Therefore, it is sufficient to prove the case where $M = M(U)$ with U/X smooth. We conclude by an argument due to Morel and Voevodsky, using Nisnevich excision as well as the fact, locally for the Zariski topology, U is étale on $\mathbf{A}^n \times X$. Then, using Nisnevich excision, we reduce to the vase where $U = \mathbf{A}^n \times X$, in which case we can provide explicit $\mathbf{A}^1$-homotopies.

Exercise 1.3.6 Show that $j_\sharp j^* M \to M \to i_* i^* M$ is not a cofiber sequence in $\underline{DM}_h(X, \Lambda)$ for an arbitrary object M.

The functor f^* restricts to a functor on DM_h, and also for $f_\sharp$ if f is smooth. Moreover, DM_h is closed under tensor product. If $i : Z \to X$ is a closed immersion, than by the cofiber sequence above we see that the functor i_* sends $DM_h(Z, \Lambda)$ to $DM_h(X, \Lambda)$.

Remark 1.3.7 By presentability, the inclusion $DM_h(X, \Lambda) \xrightarrow{i} \underline{DM}_h(X, \Lambda)$ has right adjoint ρ.

For $f : X \to Y$ we define

$$f_* : DM_h(X, \Lambda) \to DM_h(Y, \Lambda)$$

by

$$f_* M = \rho f_* i(M)$$

We can use this to describe the internal Hom as well:

$$Hom(A, B) = \sigma Hom(i(A), i(B)) .$$

Proposition 1.3.8 *For any embedding $i : Z \to X$ the functors $i_*, i_\sharp$ are both fully faithful.*

Using this and some abstract nonsense we get that i_* has a right adjoint $i^!$ and there are canonical fiber sequences

$$i_* i^! M \to M \to j_* j^* M$$

We also have a smooth base change formula and a proper base change formula:

Theorem 1.3.9 (Ayoub, Cisinski-Déglise) *For any Cartesian square of noetherian schemes*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

and for any M in $DM_h(X, \Lambda)$, if either v is separated smooth of finite type, or if f is proper, then the canonical map

$$v^* f_*(M) \to f'_* u^*(M)$$

is invertible in $DM_h(X, \Lambda)$.

The proof follows from Ayoub's axiomatic approach [Ayo07], under the additional assumption that all the maps are quasi-projective. The general case may be found in [CD19, Theorem 2.4.12].

Definition 1.3.10 (Deligne) Let $f : X \to Y$ be separated of finite type, or equivalently, by Nagata's theorem, assume that there is a relative compactification which is a factorization of f as

$$X \xrightarrow{j} \bar{X} \xrightarrow{p} Y ,$$

where j is an open embedding and p is proper. Then we define

$$f_! = p_* j_\sharp$$

Here are the main properties we will use (see [CD19]):

- The functor $f_!$ admits a right adjoint $f^!$ (because it commutes with colimits).
- There is a comparison map $f_! \to f_*$ constructed as follows. There is a map $j_\sharp \to j_*$ which corresponds by transposition to the inverse of the isomorphism from $j^* j_*$ to the identity due to the fully faithfulness of j_*. Therefore we have a map $f_! = p_* j_\sharp \to p_* j_* \cong f_*$.
- Using the proper base change formula, we can prove that push-forwards with compact support are well defined: in particular, the functor $f_!$ does not depend on the choice of the compactification of f up to isomorphism. Furthermore, if f and g are composable, there is a coherent isomorphism $f_! g_! \cong (fg)_!$.

The proof of the proper base change formula relies heavily on the following property.

Theorem 1.3.11 (Relative Purity) *If $f : X \to Y$ is smooth and separated of finite type, then*

$$f^!(M) \cong f^*(M)(d)[2d]$$

where $d = dim(X/Y)$.

The first appearance of this kind of result in a motivic context (i.e. in stable homotopy category of schemes) was in a preprint of Oliver Röndigs [Rön05]. As a matter of facts, the proof of relative purity can be made with a great level of generality, as in Ayoub's thesis [Ayo07], where we see that the only inputs are the localization theorem and $\mathbf{A}^1$-homotopy invariance. However, in our situation (where Chern classes are available), the proof can be dramatically simplified (see the proof [CD16, Theorem 4.2.6], which can easily be adapted to the context of h-sheaves). A very neat and robust proof (in equivariant stable homotopy category of schemes, but which may be seen in any context with the six operations) may be found in Hoyois' paper [Hoy17].

Remark 1.3.12 For a vector bundle $E \to X$ of rank r, we can define its *Thom space* $Th(E)$ by the cofiber sequence

$$\Lambda(E - 0) \to \Lambda(E) \to Th(E)$$

(where $E - 0$ is the complement of the zero section). Using motivic Chern classes, we can construct the Thom isomorphism

$$Th(E) \cong \Lambda(r)[2r] .$$

What is really canonical and conceptually right is

$$f^!(M) \cong f^*(M) \otimes Th(T_f).$$

We refer to Ayoub's work for more details. From this we can deduce a formula relating $f_!$ and $f_\sharp$ when f is smooth. By transposition, relative putity takes the following form.

Corollary 1.3.13 *If $f : X \to Y$ is smooth and separated of finite type then*

$$f_\sharp(M) \cong f_!(M)(d)[2d].$$

Finally, we also need the Projection Formula (see [CD19, Theorem 2.2.14]):

Proposition 1.3.14 *If $f : X \to Y$ is separated of finite type then, for any A in $DM_h(X, \Lambda)$ and any B in $DM_h(Y, \Lambda)$, there is a canonical isomorphism:*

$$f_!(A \otimes f^* B) \cong f_!(A) \otimes B\,.$$

Exercise 1.3.15

- Let $f : X \to Y$, then $f_* Hom(f^* M, N) \cong Hom(M, f_* N)$.
- For f separated of finite type we have $Hom(f_! M, N) \cong f_* Hom(M, f^! N)$.
- For f as above, $f^! Hom(M, N) \cong Hom(f^* M, f^! N)$
- For f *smooth*, $f^* Hom(M, N) \cong Hom(f^* M, f^* N)$.

A reformulation of the proper base change formula is the following.

Theorem 1.3.16 *For any pull-back square of noetherian schemes*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

with f is separated of finite type, we have $v^* f_! \cong f'_! u^*$ and $f^! v_* \cong u_* (f')^!$.

Remark 1.3.17 Given a morphism of rings of coefficients $\Lambda \to \Lambda'$, there is an obvious change of coefficients functor

$$DM_h(X, \Lambda) \to DM_h(X, \Lambda')\,, \quad M \mapsto \Lambda' \otimes_\Lambda M$$

which is symmetric monoidal and commutes with the four operations f^*, f_*, $f^!$ and $f_!$ whenever they are defined. Moreover, one can show that an object M in $DM_h(X, \mathbf{Z})$ is null if and only if $\mathbf{Q} \otimes M \cong 0$ and $\mathbf{Z}/p\mathbf{Z} \otimes M \cong 0$ for any prime number p; see [CD16, Prop. 5.4.12]. Fortunately, $DM_h(X, \Lambda)$ may be understood in more tractable terms whenever $\Lambda = \mathbf{Q}$ of Λ is finite, as we will see in the next section.

1.4 Representability theorems

1.4.1 We define *étale motivic cohomology*[3] of X with coefficients in Λ as

$$H^i_M(X, \Lambda(n)) = H^i(\mathrm{Hom}_{DM_h(X,\Lambda)}(\Lambda, \Lambda(n)))$$

for all $i, n \in \mathbf{Z}$.

[3] Also known as Lichtenbaum cohomology.

Theorem 1.4.2 (Suslin-Voevodsky, Cisinski-Déglise) *For any noetherian scheme of finite dimension X,*

$$H^i_M(X, \mathbf{Q}(n)) \cong (KH_{2n-i}(X) \otimes \mathbf{Q})^{(n)}$$

where KH is the homotopy invariant K-theory of Weibel and the "$^{(n)}$" stands for the fact that we take the intersection of the k^n-eigen-spaces of the Adams operations ψ_k for all k. For X regular, we simply have $H^i_M(X, \mathbf{Q}(n)) \cong (K_{2n-i}(X) \otimes \mathbf{Q})^{(n)}$. In particular, for X regular and $n \in \mathbf{Z}$, we have:

$$CH^n(X) \otimes \mathbf{Q} \cong H^{2n}_M(X, \mathbf{Q}(n))\,.$$

The case where X is separated and smooth of finite type over a field is due to Suslin and Voevodsky (puting together the results of [SV96] and of [VSF00]). The general case follows from [CD16, Theorem 5.2.2], using the representability theorem of KH announced in [Voe98] and proved in [Cis13]. More generally, one may recover motivically $\mathbf{Q}$-linear Chow groups of possibly singular schemes as well as Bloch's higher Chow groups as follows.

Theorem 1.4.3 (motivic cycle class) *Let $f : X \to Spec(k)$ be separated of finite type. Then*

$$H^0(\mathrm{Hom}_{DM_h(X,\mathbf{Q})}(\mathbf{Q}(n)[2n], f^!\mathbf{Q})) \cong CH_n(X) \otimes \mathbf{Q}$$

and, if X is equidimensional of dimension d, then

$$H^0(\mathrm{Hom}_{DM_h(X,\mathbf{Q})}(\mathbf{Q}(n)[i], f^!\mathbf{Q})) \cong CH^{d-n}(X, i-2n) \otimes \mathbf{Q}\,.$$

This follows from [CD15, Corollaries 8.12 and 8.13, Remark 9.7]. These representability result may be used to see how classical Grothendieck motives of smooth projective varieties over a field k may be seen in this picture: they form the full subcategory of $DM_h(\mathrm{Spec}(k), \mathbf{Q})$ whose objects are the direct factors of motives of the form $M(U)(n)$ with U smooth and projective over k and $n \in \mathbf{Z}$). The following statement is known as *rigidity theorem*

Theorem 1.4.4 (Suslin-Voevodsky, Cisinski-Déglise) *Given a locally noetherian scheme X, there is a canonical equivalence of ∞-categories*

$$DM_h(X, \Lambda) \cong D(Sh(X_{et}, \Lambda))$$

for Λ of positive invertible characteristic on X, compatible with 6-operations. In particular

$$H^i_M(X, \Lambda(j)) \cong H^i_{et}(X, \mu_n^{\otimes j} \otimes \Lambda).$$

The case where X is the spectrum of a field is essentially contained in the work of Suslin and Voevodsky [SV96]. See [CD16, Corollary 5.5.4] for the general case. We should mention that the equivalence of categories above is easy to construct. The main observation is Voevodsky's theorem 1.2.8, together with the Kummer short exact sequence induced by $t \mapsto t^n$

$$0 \to \mu_n \to \mathbf{G}_m \to \mathbf{G}_m \to 0$$

(where μ_n is the sheaf of n-th roots of unity), from which follows the identification $\Lambda(1) \cong \mu_n \otimes_{\mathbf{Z}/n\mathbf{Z}} \Lambda$, where n is the characteristic of Λ. In particular, $\Lambda(1)$ is already $\otimes$-invertible, which implies (by inspection of universal properties) that

$$\underline{DM}_h^{\mathit{eff}}(X, \Lambda) \cong \underline{DM}_h(X, \Lambda)\,.$$

On the other hand, $\underline{DM}_h^{\mathit{eff}}(X, \Lambda)$ is a full subcategory of the derived category of h-sheaves of Λ-modules. The comparison functor from $\underline{DM}_h^{\mathit{eff}}(X, \Lambda)$ to $D(Sh(X_{et}, \Lambda))$ is simply the restriction functor. The precise formulation of the previous theorem is that the composition

$$DM_h(X, \Lambda) \subset \underline{DM}_h(X, \Lambda) \cong \underline{DM}_h^{\mathit{eff}}(X, \Lambda) \to D(Sh(X_{et}, \Lambda))$$

is an equivalence of ∞-categories.

Remark 1.4.5 If $char(\Lambda) = p^i$ then one proves that $DM_h(X, \Lambda) \cong DM_h(X[\frac{1}{p}], \Lambda)$ (using the Artin-Schreier short exact sequence together with the localization property) so that we can assume that the ring of functions on X always has the characteristic of Λ invertible in it; see [CD16].

Remark 1.4.6 One can have access to $H^i_M(X, \mathbf{Z}(n))$ via the coniveau spectral sequence whose E_1 term is computed as Cousin complex, and thus gives rise to a nice and rather explicit theory of residues; see [CD16, (7.1.6.a) and Prop. 7.1.10].

2 Finiteness and Euler characteristic

2.1 Locally constructible motives

2.1.1 Recall that an object X in a tensor category C is *dualizable* (we also say *rigid*) if there exists $Y \in C$ such that $X \otimes -$ is left adjoint to $Y \otimes -$. This provides an isomorphism $Y \cong \mathcal{H}om(X, 1_C)$. In other words $Y \otimes a \cong \mathcal{H}om(X, a)$. This way, we get the evaluation map $\epsilon : Y \otimes X \to 1_C$ and as well as the co-evaluation map $\eta : 1_C \to X \otimes Y$. This exhibits the adjunction between the tensors. In particular, composing ϵ and η approriately tensored by X or Y gives the identity:

$$1_X : X \to X \otimes Y \otimes X \to X \quad \text{and} \quad 1_Y : Y \to Y \otimes X \otimes Y \to Y\,.$$

Remark 2.1.2 If $F : C \to D$ is a monoidal functor, if $x \in C$ dualizable then so is $F(x)$, and $F(x^\wedge) \cong F(x)^\wedge$. Furthermore, F also preserve internal $\mathcal{H}om$ from x, since $\mathcal{H}om(x, y) \cong x^\wedge \otimes y$ for all y.

Remark 2.1.3 If $C \in D(Sh_{et}(X, \Lambda))$ then it is dualizable if and only if it is locally constant with perfect fibers; see [CD16, Remark 6.3.27]. That means that C is dualizable if and only if the following condition holds: there is a surjective étale map $u : X' \to X$ together with a perfect complex of Λ-modules $K \in Perf(\Lambda)$ (i.e. complex of Λ-modules K which is quasi-isomorphic to a bounded complex of projective Λ-modules of finite type), and an isomorphism $K_{X'} \cong u^*(C)$ in $D(Sh_{et}(X', \Lambda))$, where $K_{X'}$ is the constant sheaf on X' associated to K.

2.1.4 Suppose $1/n \in \mathcal{O}_X$, $n = char(\Lambda) > 0$. Then $DM_h(X, \Lambda) \cong D(Sh_{et}(X, \Lambda))$. Inside it, we have the subcategory $D^b_{ctf}(X_{et}, \Lambda)$ of constructible sheaves finite tor-dimension. If there is d such that $cd(k(x)) \leq d$ for every point x of X, then it is simply the subcategory of compact objects. In general, this subcategory $D^b_{ctf}(X_{et}, \Lambda)$ is important because it is closed under the six operations. We look for correspondent in motives with arbitrary ring of coefficients Λ. We can characterise those étale sheaves by

$$\{C \in D(Sh_{et}(X, \Lambda)) \mid \exists \text{ stratification } X_i : C_{|X_i} \text{ locally constant with perfect fibers}\}$$

Namely, an object C of $D(Sh_{et}(X, \Lambda))$ is constructible of finite tor-dimension if and only if there exists a finite stratification of X by locally closed subschemes X_i together with $\phi_i : U_i \to X_i$ étale surjective for each i, and there is $K_i \in Perf(\Lambda)$ (compact objects in the derived category of Λ-modules), and an isomorphism $\phi_i^*(C_{|X_i}) \cong (K_i)_{U_i}$ in the derived category of sheaves of Λ-modules on the small étale site of U_i; see [CD16, Remark 6.3.27].

Exercise 2.1.5 (Poincaré Duality) Let $f : X \to Y$ be smooth and proper of relative dimension d. Then, if $M \in DM_h(X, \Lambda)$ is dualizable, so is $f_*(M)$ and

$$f_*(M)^\wedge \cong f_*(M^\wedge)(-d)[-2d]$$

with $M^\wedge = Hom(M, \Lambda)$ the dual of M.

Definition 2.1.6 The ∞-category $DM_{h,c}(X, \Lambda)$ of *constructible* Λ-linear étale motives over X is the smallest thick subcategory (closed under shifts, finite colimits and retracts) containing $f_\sharp(\Lambda)(n)$ for any $f : U \to X$ smooth and every $n \in \mathbf{Z}$.

The following proposition is an easy consequence of relative purity and of the proper base change formula.

Proposition 2.1.7 *The ∞-category $DM_{h,c}(X, \Lambda)$ is equal to each of the following subcategories of $DM_h(X, \Lambda)$:*

- *The smallest thick subcategory containing $f_*(\Lambda)(n)$ for $f : U \to X$ proper and $n \in \mathbf{Z}$.*
- *The smallest thick subcategory containing $f_!(\Lambda)(n)$ for $f : U \to X$ separated of finite type and $n \in \mathbf{Z}$.*

Theorem 2.1.8 (Absolute Purity) *If $i : Z \to X$ is a closed emmersion and assume that both X, Z are regular. Let $c = codim(Z, X)$. Then there is a canonical isomorphism*

$$i^!(\Lambda_X) \cong \Lambda_Z(-c)[-2c].$$

See [CD16, Theorem 5.6.2]

Remark 2.1.9 Modulo the rigidity theorem 1.4.4, the proof for the case of finite coefficients is due to Gabber and was known for a while, with two different proofs [Fuj02, ILO14] (although, in characteristic zero, this goes back to Artin in SGA 4). After formal reductions using deformation to the normal cone, one sees that, in order to prove the absolute purity theorem above, it is then sufficient to consider the case where $\Lambda = \mathbf{Q}$. The idea is then that Quillen's localization fiber sequence

$$\begin{array}{ccccc} K(Z) & \longrightarrow & K(X) & \longrightarrow & K(X-Z) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ K(Coh(Z)) & \longrightarrow & K(Coh(X)) & \longrightarrow & K(Coh(X-Z)) \end{array}$$

induces a long exact sequence which we may tensor with $\mathbf{Q}$, and Absolute purity is then proved using the representability theorem of K-theory in the motivic stable homotopy category together with a variation on the Adams-Riemann-Roch theorem.

We recall that a locally noetherian scheme X is *quasi-excellent* if the following two conditions are verified:

1. For any point $x \in X$, the completion map $\mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x}$ is regular (i.e., for any field extension K of the residue field $\kappa(x)$, the noetherian ring $K \otimes_{\kappa(x)} \hat{\mathcal{O}}_{X,x}$ is regular).
2. For any scheme of finite presentation Y over X, there is a regular dense open subscheme $U \subset Y$.

A locally noetherian scheme is *excellent* if it is quasi-excellent and universally catenary. In practice, what needs to be known is that any scheme of finite type over a quasi-excellent scheme is quasi-excellent, and $Spec(R)$ is excellent whenever R is either a field or the ring of integers of a number field (note also that noetherian complete local rings are excellent).

Theorem 2.1.10 (de Jong-Gabber [ILO14]) *Any quasi-excellent scheme is regular locally for the h-topology. In other words, for any quasi-excellent scheme X, there exists an h-covering $\{X_i \to X\}_i$ with each X_i regular. Furthermore, locally for the h-topology any nowhere dense closed subscheme of X is either empty of a divisor with normal crossings: given any nowhere dense closed subscheme $Z \subset X$, we may choose the covering above such that the pullback of Z in each X_i is either empty or a divisor with normal crossings. Even better, given a prime ℓ invertible in $\mathcal{O}_X$, we may always choose h-coverings $\{X_i \to X\}_i$ as above such that, for each point $x \in X$, there exists an i and there exists $x_i \in X_i$ such that $p_i(x_i) = x$ and such that $[k(x_i) : k(x)]$ is prime to ℓ.*

Remark 2.1.11 One can show that the category $DM_{h,c}(X, \Lambda)$ is preserved by the 6 operations. However, there is a drawback: unless we make finite cohomological dimension assumptions, the category $DM_{h,c}$ in not always a sheaf for the étale topology! Here is its étale sheafification (which can be proved to be a sheaf of ∞-categories for the h-topology).

Definition 2.1.12 A motivic sheaf M is in $DM_h(X, \Lambda)$ is *locally constructible* if there is an étale surjection $f : U \to X$ such that $f^*M \in DM_{h,c}(X, \Lambda)$.

Denote the full subcategory of locally constructible motives by $DM_{h,lc}(X, \Lambda)$.

Remark 2.1.13 If $\mathbf{Q} \subset \Lambda$, then $DM_{h,c}(X, \Lambda) = DM_{h,lc}(X, \Lambda)$ simply is the full subcategory of compact objects in $DM_h(X, \Lambda)$; see [CD16, Prop. 6.3.3].

Theorem 2.1.14 (Cisinski-Déglise) *The equivalence $DM_h(X, \Lambda) \cong D(X_{et}, \Lambda)$ restricts to an equivalence of ∞-categories*

$$DM_{h,lc}(X, \Lambda) \cong D^b_{ctf}(X, \Lambda)$$

whenever Λ is noetherian of positive characteristic n, with $\frac{1}{n} \in \mathcal{O}_X$.

See [CD16, Theorem 6.3.11].

For any morphism of noetherian schemes $f : X \to Y$, the functor f^* sends locally constructible h-motives to locally constructible h-motives, and, in the case where f is separated of finite type, so does the functor $f_!$. The theorem of de Jong-Gabber above, together with Absolute Purity, are the main ingredients in the proof of the following finiteness theorem.

Theorem 2.1.15 (Cisinski-Déglise) *The six operations preserve locally constructible h-motives, at least when restricted to separated morphisms of finite type between quasi-excellent noetherian schemes of finite dimension:*

1. *for any such scheme X and any locally constructible h-motives M and N over X, the h-motives $M \otimes N$ and $Hom(M, N)$ are locally constructible;*
2. *for any morphism of finite type $f : X \to Y$ between quasi-excellent noetherian schemes of finite dimension, the four functors f^*, f_*, $f_!$, and $f^!$ preserve the property of being locally constructible.*

See [CD16, Corollary 6.3.15].

Theorem 2.1.16 (Cisinski-Déglise) *Let X be a noetherian scheme of finite dimension, and M an object of $DM_h(X, \Lambda)$.*

1. *If M is dualizable, then it is locally constructible.*
2. *If there exists a closed immersion $i : Z \to X$ with open complement $j : U \to X$ such that $i^*(M)$ and $j^*(M)$ are locally constructible, then M is locally constructible.*
3. *If M is locally constructible over X, then there exists a dense open immersion $j : U \to X$ such that $j^*(M)$ is dualizable in $DM_{h,lc}(U)$.*

This is a reformulation of (part of) [CD16, Theorem 6.3.26].

Remark 2.1.17 In particular, an object M of $DM_h(X, \Lambda)$ is constructible if and only if there exists a finite stratification of X by locally closed subschemes X_i such that each restriction $M_{|X_i}$ is dualizable in $DM_h(X_i, \Lambda)$. This may be seen as an independence of ℓ result. Indeed, as we will recall below, there are ℓ-adic realization functors and they commute with the six functors. In particular, for each appropriate prime number ℓ, the ℓ-adic realization $R_\ell(M)$ is a constructible ℓ-adic sheaf: each restriction $R_\ell(M)_{|X_i}$ is smooth (in the language of SGA 4, 'localement constant tordu')[4], where the X_i form a stratification of X which is given independently of ℓ. Furthermore, if we apply any of the six operations to $R_\ell(M)$ in the ℓ-adic context, then we obtain an object of the from $R_\ell(N)$ for some locally constructible motive N, and thus a stratification as above relatively to $R_\ell(N)$ which does not depend on ℓ.

2.2 Integrality of traces and rationality of ζ-Functions.

2.2.1 For x a dualizable object in a tensor category C with unit object $\mathbf{1}$, we can from the trace of an endomorphism. Indeed the trace of $f : x \to x$ is the map $Tr(f) : \mathbf{1} \to \mathbf{1}$ defined as the composite bellow.

$$\mathbf{1} \xrightarrow{\text{unit}} Hom(x,x) \cong x^\wedge \otimes x \xrightarrow{1\otimes f} x^\wedge \otimes x \xrightarrow{\text{evaluation}} \mathbf{1}$$

If a functor $\Phi : C \to D$ is symmetric monoidal, then the induced map

$$\Phi : \mathrm{Hom}_C(x,x) \to \mathrm{Hom}_C(\mathbf{1},\mathbf{1})$$

preserves the formation of traces: $\Phi(Tr(f)) = Tr(\Phi(f))$.

2.2.2 If $M \in DM_{h,lc}(Spec(k),\Lambda)$ for k a field (see Def. 2.1.12), then M is dualizable. Furthermore, the unit is Λ and we can compute

$$H^0\mathrm{Hom}_{DM_{h,lc}(Spec(k),\Lambda)}(\Lambda,\Lambda) = \Lambda \otimes \mathbf{Z}[1/p]$$

where p is the exponent characteristic of k (i.e. $p = char(k)$ if $char(k) > 0$ or $p = 1$ else). For $f : M \to M$ any map in $DM_{h,lc}(Spec(k),\mathbf{Z})$, we thus have its trace

$$Tr(f) \in \mathbf{Z}[1/p]\,.$$

The *Euler characteristic* of a dualizable object M of $DM_h(Spec(k),\mathbf{Z})$ is defined as the trace of its identity:

$$\chi(M) = Tr(1_M)\,.$$

For separated k-scheme of finite type X, we define in particular

[4] It is standard terminology to call such ℓ-adic sheaves 'lisses'. This comes from Deligne's work, which is written in French. I prefer to translate into 'smooth' because this is what we do for morphisms of schemes. The reason is that this terminology comes from the fact that there are transersality conditions one can define between (motivic or ℓ-adic) sheaves and morphisms of schemes, and that a basic intuition about smoothness is that a smooth object is transverse to anything: indeed, a smooth sheaf is transverse to any morphism, while any sheaf is transverse to a smooth morphism. This why I think it is better to use the same word to express the smoothness of both sheaves and morphisms of schemes.

$$\chi_c(X) = \chi(a_!\mathbf{Z})$$

with $a : X \to Spec(k)$ the structural map.

2.2.3 Let X be a noetherian scheme and ℓ a prime number. Let $\mathbf{Z}_{(\ell)}$ be the localization of $\mathbf{Z}$ at the prime ideal (ℓ). We may identify $DM_h(X, \mathbf{Q})$ as the full subcategory of $DM_h(X, \mathbf{Z}_{(\ell)})$ whose objects are the motives M such that $M/\ell M \cong 0$, where $M/\ell M \cong \mathbf{Z}/\ell\mathbf{Z} \otimes M$ is defined via the following cofiber sequence:

$$M \xrightarrow{\ell} M \to M/\ell M\,.$$

We define

$$\hat{D}(X, \mathbf{Z}_\ell) = DM_h(X, \mathbf{Z}_{(\ell)})/DM_h(X, \mathbf{Q})\,.$$

In other words, $\hat{D}(X, \mathbf{Z}_\ell)$ is the localization (in the sense of ∞-categories) of $DM_h(X, \mathbf{Z}_{(\ell)})$ by the maps $f : M \to N$ whose cofiber is uniquely ℓ-divisible (i.e. lies in the subcategory $DM_h(X, \mathbf{Q})$). One can show that, if $\frac{1}{\ell} \in \mathcal{O}_X$, the homotopy category of $\hat{D}(X, \mathbf{Z}_\ell)$ is Ekedahl's derived category of ℓ-adic sheaves on the small étale site of X. In fact, as explained in [CD16, Prop. 7.2.21] (although in the language of model categories), the rigidity theorem 1.4.4 may be interpreted as an equivalence of ∞-categories of the form:

$$\hat{D}(X, \mathbf{Z}_\ell) \cong \lim_n D(X_{et}, \mathbf{Z}/\ell^n\mathbf{Z})$$

(here, the limit is taken in the ∞-categories of ∞-categories). We thus have a canonical *ℓ-adic realization functor*

$$R_\ell : DM_h(X, \mathbf{Z}) \to \lim_n D(X_{et}, \mathbf{Z}/\ell^n\mathbf{Z})$$

which sends a motive M to $M \otimes \mathbf{Z}_{(\ell)}$, seen in the Verdier quotient $\hat{D}(X, \mathbf{Z}_\ell)$. We observe that there is a unique way to define the six operations on $\hat{D}(X, \mathbf{Z}_\ell)$ in such a way that the ℓ-adic realization functor commutes with them. In particular, there is a symmetric monoidal structure on $\hat{D}(X, \mathbf{Z}_\ell)$.

Classically, one defines $D^b_c(X_{et}, \mathbf{Z}_\ell)$ as the full subcategory of $\lim_n D(X_{et}, \mathbf{Z}/\ell^n\mathbf{Z})$ whose objects are the ℓ-adic systems $(\mathcal{F}_n)$ such that each $\mathcal{F}_n$ belongs to the subcategory $D^b_{ctf}(X_{et}, \mathbf{Z}/\ell^n\mathbf{Z})$ (see 2.1.4). Furthermore, an ℓ-adic system $(\mathcal{F}_n)$ is dualizable is and only if $\mathcal{F}_1$ is dualizable in $D^b_{ctf}(X_{et}, \mathbf{Z}/\ell\mathbf{Z})$: this is due to the fact, that, by definition, the canonical functor

$$\hat{D}(X, \mathbf{Z}_\ell) \to D(X_{et}, \mathbf{Z}/\ell\mathbf{Z})$$

is symmetric monoidal, conservative, and commutes with the formation of internal Hom's. In other words, $D^b_c(X_{et}, \mathbf{Z}_\ell)$ may be identified with the full subcategory of $\hat{D}(X, \mathbf{Z}_\ell)$ whose objects are those $\mathcal{F}$ such that there exists a finite stratification by locally closed subschemes $X_i \subset X$ such that each restriction $\mathcal{F}_{|X_i}$ is dualizable in $\hat{D}(X_i, \mathbf{Z}_\ell)$. We thus have a canonical equivalence of ∞-categories:

$$D^b_c(X_{et}, \mathbf{Z}_\ell) \cong \lim_n D^b_{ctf}(X_{et}, \mathbf{Z}/\ell^n\mathbf{Z}) .$$

This implies right away that the six operations restrict to $D^b_c(X_{et}, \mathbf{Z}_\ell)$ (if we consider quasi-excellent schemes only), and, by vitue of Theorem 2.1.14 that we have an ℓ-adic realization functor

$$R_\ell : DM_{h,lc}(X, \mathbf{Z}) \to D^b_c(X_{et}, \mathbf{Z}_\ell)$$

which commute with the six operations. For a scheme X with structural map $a : X \to \mathrm{Spec}(k)$ separated and of finite type, the motive of X is $M(X) = a_! a^!(\mathbf{Z})$. This is a dualizable object with dual $a_*(\mathbf{Z})$. Hence

$$R_\ell(M(X)^\wedge(n)) = a_*(\mathbf{Z}_\ell(n)) = R\Gamma(X_{et}, \mathbf{Z}_\ell(n))$$

is a dualizable object in $D^b_c(\mathrm{Spec}(k)_{et}, \mathbf{Z}_\ell)$. For k separably closed, the latter category simply is the bounded derived category of $\mathbf{Z}_\ell$-modules of finite type, and this proves in particular that ℓ-adic cohomology

$$H^i_{et}(X, \mathbf{Z}_\ell(n)) = H^i(R_\ell(M(X)^\wedge(n)))$$

is of finite type as a $\mathbf{Z}_\ell$-module for all i (and trivial for all but finitely many i's). Similarly, the ℓ-adic realization of $a_!(\mathbf{Z})$ gives ℓ-adic cohomology with compact support

$$H^i_{et,c}(X, \mathbf{Z}_\ell(n)) = H^i(R_\ell(a_!(\mathbf{Z})(n))) .$$

2.2.4 In particular, for any field k of characteristic prime to ℓ, we have a symmetric monoidal functor

$$R_\ell : DM_{h,lc}(k, \mathbf{Z}) \to D^b_c(k, \mathbf{Z}_\ell)$$

inducing the map of rings

$$R_\ell : \mathbf{Z}[1/p] \cong H^0\mathrm{Hom}_{DM_{h,lc}(k,\mathbf{Z})}(\mathbf{Z}, \mathbf{Z}) \to H^0\mathrm{Hom}_{D^b_c(k,\mathbf{Z}_\ell)}(\mathbf{Z}_\ell, \mathbf{Z}_\ell) \cong \mathbf{Z}_\ell .$$

Therefore, for an endomorphism $f : M \to M$ we have $Tr(f) \in \mathbf{Z}[1/p]$ sent to the ℓ-adic number $Tr(R_\ell(f)) \in \mathbf{Z}_\ell$. We thus get:

Corollary 2.2.5 *The ℓ-adic trace $Tr(R_\ell(f)) \in \mathbf{Z}[1/p]$ and is independent of ℓ.*

Remark 2.2.6 If k is separably closed, then $D^b_c(k, \mathbf{Z}_\ell)$ simply is the derived category of $\mathbf{Z}_\ell$-modules of finite type, and we have

$$Tr(R_\ell(f)) = \sum_i (-1)^i Tr(H^i R_\ell(f) : H^i R_\ell(M) \to H^i R_\ell(M))$$

where each $Tr(H^i R_\ell(f))$ can be computed in the usual way in terms of traces of matrices. If k is not separably closed, we can always choose a separable closure $\bar{k}$ and observe that pulling back along the map $Spec(\bar{k}) \to Spec(k)$ is a symmetric monoidal functor which commutes with the ℓ-adic realization functor. This can actually be used to prove that the Euler characteristic is always an integer (as opposed to a rational number in $\mathbf{Z}[1/p]$): if $f = 1_M$ is the identity, the trace of $R_\ell(f)$ can be computed as an alternating sum of ranks of $\mathbf{Z}_\ell$-modules of finite type.

Corollary 2.2.7 *For any dualizable object M in $DM_h(k, \mathbf{Z})$, we have $\chi(M) \in \mathbf{Z}$.*

2.2.8 Let A be a ring. A function $f : X \to A$ from a topological space to a ring is constructible if there is a finite stratification of X by locally closed X_i such that each $f|_{X_i}$ is constant. We denote by $C(X, A)$ the ring of constructible functions with values in A on X. For a scheme X, we define $C(X, A) = C(|X|, A)$, where $|X|$ denotes the topological space underlying X.

2.2.9 Recall that, for a stable ∞-category C, we have its Grothendieck group $K_0(C)$: the free monoid generated by isomorphism classes $[x]$ of objects x of C, modulo the relations $[x] = [x'] + [x'']$ for each cofiber sequence $x' \to x \to x''$. In particular, we have the relations $0 = [0]$ and $[x] + [y] = [x \oplus y]$. This monoid turns out to be an abelian group with $-[x] = [\Sigma(x)]$. If ever C is symmetric monoidal, then $K_0(C)$ inherits a commutative ring structure with multiplication $[x][y] = [x \otimes y]$.

2.2.10 We have the Euler characteristic map $DM_{h,lc}(X, \mathbf{Z}) \xrightarrow{\chi} C(X, \mathbf{Z})$. It is defined by $\chi(M)(x) = \chi(x^*M)$, where the point x is seen as a map $x : Spec(\kappa(x)) \to X$. Recall that if $M \in DM_h(X, \Lambda)$ is locally constructible then there is $U \subseteq X$ open and dense such that $M|_{(X-U)_{red}}$ is locally constructible and $M|_U$ is dualizable. Therefore, by noetherian induction, we see that $\chi(M) : |X| \to \mathbf{Z}$ is a constructible function indeed. For any cofiber sequence of dualizable objects

$$M' \to M \to M'' ,$$

we have

$$\chi(M) = \chi(M') + \chi(M'') .$$

Since $\chi(M \otimes N) = \chi(M)\chi(N)$, we have a morphism of rings:

$$\chi : K_0(DM_{h,lc}(X, \mathbf{Z})) \to C(X, \mathbf{Z}) ,$$

and we have a commutative triangle:

$$\begin{array}{ccc} K_0(DM_{h,lc}(X, \mathbf{Z})) & \xrightarrow{R_\ell} & K_0(D^b_c(X_{et}, \mathbf{Z}_\ell)) \\ & {\scriptstyle\chi}\searrow \quad \swarrow{\scriptstyle\chi} & \\ & C(X, \mathbf{Z}) & \end{array}$$

2.2.11 Given a stable ∞-category C, there is the full subcategory C_{tors} which consists of objects x such that there exists an integer n such that $n.1_x \cong 0$. One checks that C_{tors} is a thick subcategory of C and one defines the Verdier quotient $C \otimes \mathbf{Q} = C/C_{tors}$. All this is a fancy way to say that one defines $C \otimes \mathbf{Q}$ as the ∞-category with the same set of objects as C, such that $\pi_0 Map_C(x, y) \otimes \mathbf{Q} = \pi_0 Map_{C\otimes\mathbf{Q}}(x, y)$ for all x and y. This is how one defines ℓ-adic sheaves:

$$D^b_c(X_{et}, \mathbf{Q}_\ell) = D^b_c(X_{et}, \mathbf{Z}_\ell) \otimes \mathbf{Q}\,.$$

When it comes to motives, we can prove that, when X is noetherian of finite dimension, the canonical functor

$$DM_{h,lc}(X, \mathbf{Z}) \otimes \mathbf{Q} \to DM_{h,lc}(X, \mathbf{Q})$$

is fully faithful and almost an equivalence: a Morita equivalence. Since $DM_{h,lc}(X, \mathbf{Q})$ is idempotent complete, that means that any $\mathbf{Q}$-linear locally constructible motive is a direct factor of a $\mathbf{Z}$-linear one. Furthermore, one checks that $D^b_c(X_{et}, \mathbf{Q}_\ell)$ is idempotent complete (because it has a bounded t-structure), so that we get a $\mathbf{Q}$-linear ℓ-adic realization functor:

$$R_\ell : DM_{h,lc}(X, \mathbf{Q}) \to D^b_c(X_{et}, \mathbf{Q}_\ell)$$

which is completely determined by the fact that the following square commutes.

$$\begin{array}{ccc} DM_{h,lc}(X, \mathbf{Z}) & \xrightarrow{R_\ell} & D^b_c(X_{et}, \mathbf{Z}_\ell) \\ \downarrow & & \downarrow \\ DM_{h,lc}(X, \mathbf{Q}) & \xrightarrow{R_\ell} & D^b_c(X_{et}, \mathbf{Q}_\ell) \end{array}$$

The $\mathbf{Q}$-linear ℓ-adic realization functor commutes with the six operations if we restrict ourselves to quasi-excellent schemes over $\mathbf{Z}[1/\ell]$; see [CD16, 7.2.24].

We may see these realization functors as a categorified version of cycle class maps. Indeed, in view of the representability results such as Theorem 1.4.3 , they induce the classical cycle class maps in ℓ-adic cohomomology: for a field k and a separated morphism of finite type $a : X \to \operatorname{Spec}(k)$, we have

$$\begin{array}{ccc} H^0(\operatorname{Hom}_{DM_h(X,\mathbf{Q})}(\mathbf{Q}(n)[2n], a^!\mathbf{Q})) & \xrightarrow{R_\ell} & H^0(\operatorname{Hom}_{D^b_c(X_{et},\mathbf{Q}_\ell)}(\mathbf{Q}_\ell(n)[2n], a^!\mathbf{Q}_\ell)) \\ \| \wr & & \| \wr \\ CH_n(X) \otimes \mathbf{Q} & \dashrightarrow & H^{2n}_c(X_{et}, \mathbf{Q}_\ell(n))^\wedge \end{array}$$

If X is regular (e.g. smooth) this gives by Poincaré duality the cycle class map:

$$CH^n(X) \to H^{2n}(X_{et}, \mathbf{Q}_\ell(n))\,.$$

One can lift these cycle class maps to integral coefficients using similar arguments from cdh-motives; see [CD15].

Theorem 2.2.12 *There is a canonical exact sequence of the form:*

$$K_0(DM_{h,lc}(X, \mathbf{Z})_{tors}) \to K_0(DM_{h,lc}(X, \mathbf{Z})) \to K_0(DM_{h,lc}(X, \mathbf{Q})) \to 0\,.$$

Proof Let $DM_h(X, \mathbf{Z})'$ be the smallest localizing subcategory of $DM_h(X, \mathbf{Z})$ generated by $DM_{h,lc}(X, \mathbf{Z})_{tors}$. We also define $D(X_{et}, \mathbf{Z})'$ as the smallest localizing subcateory of $D(X_{et}, \mathbf{Z})$ generated by objects of the form $j_!(\mathcal{F})$, where $j : U \to X$ is a dense open immersion and $\mathcal{F}$ is bounded with constructible cohomology sheaf, such that there is a prime p with the following two properties:

- $p.1_{\mathcal{F}} = 0$;
- p is invertible in $\mathcal{O}_U$.

Then a variant of the rigidity theorem 1.4.4 (together with remark 1.4.5) gives an equivalence of ∞-categories:

$$DM_h(X, \mathbf{Z})' \cong D(X_{et}, \mathbf{Z})' .$$

One then checks that the t-structure on $D(X_{et}, \mathbf{Z})'$ induces a bounded t-structure on $DM_{h,lc}(X, \mathbf{Z})_{tors}$ (with noetherian heart, since we get a Serre subcategory of constructible étale sheaves of abelian groups on X_{et}). Using the basic properties of non-connective K-theory [Sch06, CT11, BGT13], we see that we have an exact sequence

$$K_0(DM_{lc}(X)_{tors}) \to K_0(DM_{lc}(X)) \to K_0(DM_{lc}(X)_{\mathbf{Q}}) \to K_{-1}(DM_{lc}(X)_{tors}) ,$$

where $DM_{lc}(X) = DM_{h,lc}(X, \mathbf{Z})$ and $DM_{lc}(X)_{\mathbf{Q}} = DM_{h,lc}(X, \mathbf{Q})$. By virtue of a theorem of Antieau, Gepner and Heller [AGH19], the existence of a bounded t-structure with noetherian heart implies that $K_{-i}(DM_{h,lc}(X, \mathbf{Z})_{tors}) = 0$ for all $i > 0$. □

Here is a rather concrete consequence (since $\chi(M) = 0$ for M in $DM_{h,lc}(X, \mathbf{Z})_{tors}$).

Corollary 2.2.13 *For any M in $DM_{h,lc}(X, \mathbf{Q})$, there exists M_0 in $DM_{h,lc}(X, \mathbf{Z})$, such that, for any point x in X, we have $\chi(x^*M) = \chi(x^*M_0)$.*

Remark 2.2.14 It is conjectured that there is a (nice) bounded t-structure on $DM_{h,lc}(X, \mathbf{Q})$. Since $DM_{h,lc}(X, \mathbf{Z})_{tors})$ has a bounded t-structure, this would imply the existence of a bounded t-structure on $DM_{h,lc}(X, \mathbf{Z})$, which, in turns would imply the vanishing of $K_{-1}(DM_{h,lc}(X, \mathbf{Z}))$ (see [AGH19]). Such a vanishing would mean that all Verdier quotients of $DM_{h,lc}(X, \mathbf{Z})$ would be idempotent-complete (see [Sch06, Remark 1 p. 103]). In particular, we would have an equivalence of ∞-categories $DM_{h,lc}(X, \mathbf{Z}) \otimes \mathbf{Q} = DM_{h,lc}(X, \mathbf{Q})$. The previous proposition is a virtual approximation of this expected equivalence.

2.2.15 Let R be a ring and let $W(R) = 1 + R[[t]]$ the set of power series with coefficients in R and leading term equal to 1. It has an abelian group structure defined by the multiplication of power series. And it has a unique multiplication $*$ such that $(1 + at) * (1 + bt) = 1 + abt$, turning $W(R)$ into a commutative ring: the ring of Witt vectors. We also have the subset $W(R)_{rat} \subseteq W(R)$ of rational functions, which one can prove to be a subring. Given a (stable) ∞-category C, we define

$$C^{\mathbf{N}} = \{\text{objects of } C \text{ equipped with an endomorphism}\} .$$

This is again a stable ∞-category. For $C = Perf(R)$ the ∞-category of perfect complexes on the ring R, we have an exact sequence

$$0 \to K_0(Perf(R)) \to K_0(Perf(R)^{\mathbf{N}}) \to W(R)_{rat} \to 0$$

where the first map sends a perfect complex of R-modules M to the class of M equipped with the zero map $0 : M \to M$, while the second maps sends $f : M \to M$ to $\det(1 - tf)$ (it is sufficient to check that these maps are well defined when M is a projective module of finite type, since these generate the K-groups); see [Alm78]. The first map identifies $K_0(R)$ with an ideal of $K_0(Perf(R)^{\mathbf{N}})$ so that we really get an isomorphism of commutative rings:

$$K_0(Perf(R)^{\mathbf{N}})/K_0(Perf(R)) \cong W(R)_{rat} \, .$$

2.2.16 Let k be a field with a given algebraic closure $\bar{k}$, as well as prime number ℓ which is distinct from the characteristic of k. We observe that $D^b_c(\bar{k}, \mathbf{Q}_\ell)$ simply is the bounded derived category of complexes of finite dimensional $\mathbf{Q}_\ell$-vector spaces. We thus have a symmetric monoidal realization functor

$$DM_{h,lc}(k, \mathbf{Q}) \to D^b_c(k, \mathbf{Q}_\ell) \to D^b_c(\bar{k}, \mathbf{Q}_\ell) \cong Perf(\mathbf{Q}_\ell) \, .$$

This induces a functor

$$DM_{h,lc}(k, \mathbf{Q})^{\mathbf{N}} \to Perf(\mathbf{Q}_\ell)^{\mathbf{N}} \, ,$$

and thus a map

$$K_0(DM_{h,lc}(k, \mathbf{Q})^{\mathbf{N}}) \to K_0(Perf(\mathbf{Q}_\ell)^{\mathbf{N}})$$

inducing a ring homomorphism, the ℓ-adic Zeta function

$$Z_\ell : K_0(DM_{h,lc}(k, \mathbf{Q})^{\mathbf{N}})/K_0(DM_{h,lc}(k, \mathbf{Q})) \to W(\mathbf{Q}_\ell)_{rat} \subseteq 1 + \mathbf{Q}_\ell[[t]] \, .$$

On the other hand, for an endomorphism $f : M \to M$ in $DM_{h,lc}(X, \mathbf{Q})$, one defines its motivic Zeta function as follows

$$Z(M, f) = \exp\Big(\sum_{n \geq 1} Tr(f^n) \frac{t^n}{n}\Big) \in 1 + \mathbf{Q}[[t]] \, .$$

Basic linear algebra show that $Z(M, f) = Z_\ell(M, f)$ (see [Alm78]). In particular, we see that the ℓ-adic Zeta function $Z_\ell(M, f)$ has rational coefficients and is independent of ℓ, while the motivic Zeta function $Z(M, f)$ is rational. In other words, we get a morphism of rings

$$Z : K_0(DM_{h,lc}(X, \mathbf{Q})^{\mathbf{N}})/K_0(DM_{h,lc}(X, \mathbf{Q})) \to W(\mathbf{Q})_{rat} \subset W(\mathbf{Q}) \, .$$

Concretely, if there is a cofiber sequence of motivic sheaves equipped with endomorphisms in the stable ∞-category $DM_{h,lc}(X, \mathbf{Q})^{\mathbf{N}}$

$$(M', f') \to (M, f) \to (M'', f''),$$

then

$$Z(M, f)(t) = Z(M', f')(t) \cdot Z(M'', f'')(t)$$

holds in $\mathbf{Q}[[t]]$. And for two motivic sheaves equipped with endomorphisms (M, f) and (M', f') in $DM_{h,lc}(X, \mathbf{Q})^{\mathbf{N}}$, there is

$$Z(M \otimes M', f \otimes f') = Z(M, f) * Z(M', f')$$

where $*$ denotes the multiplication in the big ring of Witt vectors $W(\mathbf{Q})$.

2.2.17 Take $k = \mathbf{F}_q$ a finite field and let $M_0 \in DM_{h,lc}(k, \mathbf{Q})$, with $M = p^* M_0$, $p : spec(\bar{k}) \to spec(k)$. Let $F : M \to M$ be the induced Frobenius. We define the *Riemann-Weil Zeta function* of M_0 as:

$$\zeta(M_0, s) = Z(M, F)(t), \quad t = p^{-s}.$$

The fact that the assignment $Z(-, F)$ defines a morphism of rings with values in $W(\mathbf{Q})$ can be used to compute explicitely the Zeta function of many basic schemes such as $\mathbf{P}^n$ of $(\mathbf{G}_m)^r$; see [Ram15, Remark 2.2] for instance.

2.3 Grothendieck-Verdier duality

2.3.1 Take S be a quasi-excellent regular scheme. We choose a $\otimes$-invertible object I_S in $DM_h(S, \Lambda)$ (e.g. $I_S = \mathbf{Z}(d)[2d]$, where d is the Krull dimension of S). For $a : X \to S$ separated of finite type, we define $I_X = a^! I_S$.

Define $\mathbf{D}_X : DM_h(X, \Lambda)^{op} \to DM_h(X, \Lambda)$ by

$$\mathbf{D}_X(M) = \mathcal{H}om(M, I_X).$$

We will sometimes write $\mathbf{D}(M) = \mathbf{D}_X(M)$.

Theorem 2.3.2 *For M a locally constructible motivic sheaf over X, the canonical map $M \to \mathbf{D}_X \mathbf{D}_X(M)$ is an equivalence.*

There is a proof in the literature under the additional assumption that S is of finite type over an excellent scheme of dimension ≤ 2 (see [CD19, CD16]). But there is in fact a proof which avoids this extra hypothesis using higher categories. Here is a sketch.

Proof The formation of the Verdier dual is compatible with pulling back along an étale map. We may thus assume that M is constructible. The full subcategory of those M's such that the biduality map of the theorem is invertible is thick. Therefore, we may assume that $M = M(U)$ for some smooth X-scheme U. In particular, we may assume that $M = \Lambda \otimes \Sigma^\infty \mathbf{Z}(U)$. It is thus sufficient to prove the case where

$\Lambda = \mathbf{Z}$. By standard arguments, we see that is is sufficient to prove the case where Λ is finite or $\Lambda = \mathbf{Q}$. Such duality theorem is a result of Gabber [ILO14] for the derived category of sheaves on the small étale site of X with coefficients in Λ of positive characteristic with n invertible in $\mathcal{O}_X$. By Theorem 1.4.4 and Remark 1.4.5, this settles the case where Λ is finite. It remains to prove the case where $\Lambda = \mathbf{Q}$. We will first prove the following statement. For each separated morphism of finite type $a : X \to S$, and each integer n, the natural map

$$\mathrm{Hom}_{DM_h(X,\mathbf{Q})}(\mathbf{Q}, \mathbf{Q}(n)) \to \mathrm{Hom}_{DM_h(X,\mathbf{Q})}(I_X, I_X(n))$$

is invertible in $D(\mathbf{Q})$ (this is the map obtained by applying the global section functor $\mathrm{Hom}(\mathbf{Q}, -)$ to the unit map $\mathbf{Q} \to \mathit{Hom}(I_X, I_X)$). We observe that we may see this map as a morphism of presheaves of complexes of $\mathbf{Q}$-vector spaces

$$E \to F$$

where $E(X) = \mathrm{Hom}_{DM_h(X,\mathbf{Q})}(\mathbf{Q}, \mathbf{Q}(n))$ and $F(X) = \mathrm{Hom}_{DM_h(X,\mathbf{Q})}(I_X, I_X(n))$. For a morphism of S-schemes $f : X \to Y$, the induced map $E(Y) \to E(X)$ is induced by the functor f^*, while the induced map $F(Y) \to F(X)$ is induced by the functor $f^!$ (and the fact that $f^!(I_Y) \cong I_X$).[5] Now, we observe that both E and F are in fact h-sheaves of complexes of $\mathbf{Q}$-vector spaces. Indeed, using [CD19, Proposition 3.3.4], we see that E and F satisfy Nisnevich excision and thus are Nisnevich sheaves. On the other hand, one can also characterise h-descent for $\mathbf{Q}$-linear Nisnevich sheaves by suitable excision properties [CD19, Theorem 3.3.24]. Such properties for E and F follow right away from [CD19, Theorem 14.3.7 and Remark 14.3.38], which proves the property of h-descent for E and F. By virtue of Theorem 2.1.10, it is sufficient to prove that $E(X) \cong F(X)$ for X regular and affine. In particular, $a : X \to S$ factors through a closed immersion $i : X \to \mathbf{A}^n \times S$. By relative purity, we have

$$I_{\mathbf{A}^n \times S} \cong p^*(I_S)(n)[2n]$$

and thus $I_{\mathbf{A}^n \times S}$ is $\otimes$-invertible (where $p : \mathbf{A}^n \times S \to S$ is the second projection). This implies that

$$I_X \cong i^!(I_{\mathbf{A}^n \times S}) \cong i^!(\mathbf{Q}) \otimes i^*(I_{\mathbf{A}^n \times S})$$

(Hint: use the fact that $i^! \mathit{Hom}(A, B) \cong \mathit{Hom}(i^* A, i^! B)$). By Absolute Purity, we have $i^! \mathbf{Q} \cong \mathbf{Q}(-c)[-2c]$, where c is the codimenion of i. In particular, the object I_X is $\otimes$-invertible, and thus the unit map $\mathbf{Q} \to \mathit{Hom}(I_X, I_X)$ is invertible. This implies that the map $E(X) \to F(X)$ is invertible as well.

We will now prove that the unit map

$$\mathbf{Q} \to \mathit{Hom}(I_X, I_X)$$

[5] This is where ∞-category theory appears seriously: proving that the construction $f \mapsto f^!$ actually defines a presheaf is a highly non-trivial homotopy coherence proplem. Such construction is explained in [Rob14, Chapter 10], using the general results of [LZ15, LZ17].

is invertible in $DM_h(X, \mathbf{Q})$ for any separated S-scheme of finite type X. Equivalently, we have to prove that, for any smooth X-scheme U and any integer n, the induced map

$$\mathrm{Hom}_{DM_h(X,\mathbf{Q})}(M(U), \mathbf{Q}(n)) \to \mathrm{Hom}_{DM_h(X,\mathbf{Q})}(M(U), \mathit{Hom}(I_X, I_X)(n))$$

is invertible in $D(\mathbf{Q})$. But we have

$$\mathrm{Hom}(f_\sharp \mathbf{Q}, \mathbf{Q}(n)) \cong \mathrm{Hom}(\mathbf{Q}, \mathbf{Q}(n)) = E(U)$$

with a smooth structural map $f : U \to X$, and

$$\begin{aligned}\mathrm{Hom}(f_\sharp \mathbf{Q}, \mathit{Hom}(I_X, I_X)(n)) &\cong \mathrm{Hom}(\mathbf{Q}, f^* \mathit{Hom}(I_X, I_X)(n)) \\ &\cong \mathrm{Hom}(\mathbf{Q}, \mathit{Hom}(f^* I_X, f^* I_X)(n)) \\ &\cong \mathrm{Hom}(\mathbf{Q}, \mathit{Hom}(f^! I_X(-d)[-2d], f^! I_X(-d)[-2d])(n)) \\ &\cong \mathrm{Hom}(\mathbf{Q}, \mathit{Hom}(f^! I_X, f^! I_X)(n)) \\ &\cong \mathrm{Hom}(\mathbf{Q}, \mathit{Hom}(I_U, I_U)(n)) = F(U) .\end{aligned}$$

In other words, we just have to check that the map $E(U) \to F(U)$ is invertible, which we already know.

Finally, we can prove that the canonical map $M \to \mathbf{D}_X \mathbf{D}_X(M)$ is invertible. As already explained at the beginning of the proof, it is sufficient to prove this when M is constructible. By virtue of Proposition 2.1.7, it is sufficient to prove the case where $M = f_*(\mathbf{Q})$, for $f : Y \to X$ a proper map. We have:

$$\begin{aligned}\mathbf{D}_X f_* \mathbf{Q} &= \mathit{Hom}(f_* \mathbf{Q}, I_X) \\ &\cong f_* \mathit{Hom}(\mathbf{Q}, f^! I_X) \\ &\cong f_* f^! I_X \\ &\cong f_* I_Y .\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbf{D}_X \mathbf{D}_X(M) &\cong \mathbf{D}_X f_* I_Y \\ &\cong \mathit{Hom}(f_* I_Y, I_X) \\ &\cong f_* \mathit{Hom}(I_Y, f^! I_X) \\ &\cong f_* \mathit{Hom}(I_Y, I_Y) \\ &\cong f_* \mathbf{Q} = M ,\end{aligned}$$

and this ends the proof. □

Corollary 2.3.3 *For locally constructible motives and f a morphism between separated S-schemes of finite type, we have:*

$$\begin{aligned}\mathbf{D}f_* &\cong f_!\mathbf{D}\\ \mathbf{D}f_! &\cong f_*\mathbf{D}\\ \mathbf{D}f^! &\cong f^*\mathbf{D}\\ \mathbf{D}f^* &\cong f^!\mathbf{D}\,.\end{aligned}$$

(The proof is by showing tautologically the second one and the fourth one, and then deduce the other two using that $\mathbf{D}$ is an involution.)

Proposition 2.3.4 *For any M and N in $DM_h(X,\Lambda)$, if N is locally constructible, then*

$$\mathbf{D}(M\otimes\mathbf{D}N)\cong \mathit{Hom}(M,N)\,.$$

Proof We construct a canonical comparison morphism:

$$\mathit{Hom}(M,N)\to\mathbf{D}(M\otimes\mathbf{D}N)\,.$$

By transposition, it corresponds to a map

$$M\otimes \mathit{Hom}(M,N)\otimes\mathbf{D}(N)\to I_X\,.$$

Such a map is induced by the evaluation maps

$$M\otimes\mathit{Hom}(M,N)\to N\quad\text{and}\quad N\otimes\mathbf{D}(N)\to I_X\,.$$

For N fixed, the class of M's such that this map is invertible is closed under colimits. Therefore, we reduce the question to the case where $M=f_\sharp\Lambda$ for $f:X\to S$ a smooth map of dimension d. In that case, we have

$$\mathit{Hom}(M,N)\cong f_*f^*(N),$$

while

$$\begin{aligned}\mathbf{D}(M\otimes\mathbf{D}N) &\cong \mathbf{D}(f_!f^*(\Lambda(-d)[-2d])\otimes\mathbf{D}N)\\ &\cong \mathbf{D}(f_!f^*(\Lambda)\otimes\mathbf{D}N)(d)[2d]\\ &\cong \mathbf{D}(f_!f^*(\mathbf{D}N))(d)[2d]\\ &\cong f_*f^!(\mathbf{D}\mathbf{D}N)(d)[2d]\\ &\cong f_*f^*N\,,\end{aligned}$$

which ends the proof. □

Corollary 2.3.5 *For M and N locally constructible on X, we have:*

$$M\otimes N\cong\mathbf{D}\mathit{Hom}(M,\mathbf{D}N)\,.$$

2.4 Generic base change: a motivic variation on Deligne's proof

2.4.1 The following statement, is a motivic analogue of Deligne's generic base change theorem for torsion étale sheaves [Del77, Th. Finitude, 1.9]. The proof follows essentially the same pattern as Deligne's original argument, except that locally constant sheaves are replaced by dualizable objects, as we will explain below. We will write $DM_h(X) = DM_h(X, \Lambda)$ for some fixed choice of coefficient ring Λ.

Theorem 2.4.2 (Motivic generic base change formula) *Let $f : X \to Y$ be a morphism between separated schemes of finite type over a noetherian base scheme S. Let M be a locally constructible h-motive on X. Then there is a dense subscheme $U \subset S$ such that the formation of $f_*(M)$ is compatible with any base-change which factors through U. Namely, for each $w : S' \to S$ factoring through U we have*

$$v^* f_* M \cong f'_* u^* M$$

where

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \\ \downarrow & & \downarrow \\ S' & \xrightarrow{w} & S \end{array}$$

is the associated pull-back diagram.

Remark 2.4.3 The motivic generic base change formula is also a kind of independence of ℓ result for each prime ℓ so that the ℓ-adic realization is defined, the formation of $f_* R_\ell(M) \cong R_\ell(f_* M)$ is compatible with any base change over $U \subset S$, where U is a dense open subscheme which is given independently of ℓ.

The first step in the proof of Theorem 2.4.2 is to find sufficient conditions for the formation a direct image to be compatible with arbitrary base change.

Proposition 2.4.4 *Let $f : X \to S$ be a smooth morphism of finite type between noetherian schemes, and let us consider a locally constructible h-motive M over X. Assume that M is dualizable in $DM_{h,lc}(X)$ and that the direct image with compact support of its dual $f_!(M^\wedge)$ is dualizable as well in $DM_{h,lc}(S)$. Then $f_*(M)$ is dualizable (in particular, locally constructible), and, for any pullback square of the form*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{v} & S \end{array}$$

the morphism f' is smooth, the pullback $u^*(M)$ is dualizable, so is $f'_!(u^*(M)^\wedge)$, and, furthermore, the canonical base change map $v^* f_*(M) \to f'_* u^*(M)$ is invertible.

Proof If d denotes the relative dimension of X over S (seen as a locally constant function over S), we have:

$$\begin{aligned} f_*(M) &\simeq f_* Hom(M^\wedge, \Lambda) \\ &\simeq f_* Hom(M^\wedge, f^!\Lambda)(-d)[-2d] \\ &\simeq Hom(f_!(M^\wedge), \Lambda)(-d)[-2d] \\ &\simeq (f_!(M^\wedge))^\wedge(-d)[-2d] \end{aligned}$$

(where the dual of a dualizable object A is denoted by $A^\wedge$). Remark that pullback functors v^* are symmetric monoidal and thus preserve dualizable objects as well as the formation of their duals. Therefore, for any pullback square of the form

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{v} & S \end{array}$$

we have that f' is smooth of relative dimension d, that $u^*(M)$ is dualizable with dual $u^*(M)^\wedge \simeq u^*(M^\wedge)$, and:

$$\begin{aligned} v^* f_*(M) &\simeq v^*(f_!(M^\wedge))^\wedge(-d)[-2d] \\ &\simeq (v^* f_!(M^\wedge))^\wedge(-d)[-2d] \\ &\simeq (f'_! u^*(M^\wedge))^\wedge(-d)[-2d] \\ &\simeq (f'_!(u^*(M)^\wedge))^\wedge(-d)[-2d] \end{aligned}$$

This also shows that $f'_!(u^*(M)^\wedge)$ is dualizable and thus that there is a canonical isomorphism

$$(f'_!(u^*(M)^\wedge))^\wedge(-d)[-2d] \simeq f'_*(u^*(M)) .$$

We deduce right away from there that the canonical base change map $v^* f_*(M) \to f'_*(u^*(M))$ is invertible. □

Remark 2.4.5 In the preceding proposition, we did not use any particular property of $DM_{h,lc}$: the statement and its proof hold in any context in which we have the six operations (more precisely, we mainly used the relative purity theorem as well as the proper base change theorem).

In order to prove Theorem 2.4.2 in general, we need to verify the following property of h-motives.

Proposition 2.4.6 *Let S be a noetherian scheme of finite dimension, and $f : Y \to S$ a quasi-finite morphism of finite type. The functors $f_! : DM_h(X) \to DM_h(S)$ and $f_* : DM_h(X) \to DM_h(S)$ are conservative.*

Proof If f is an immersion, then $f_!$ and f_* are fully faithful, hence conservative. Since the composition of two conservative functors is conservative, Zariski's Main

Theorem implies that it is sufficient to prove the case where f is finite. In this case, since the formation of $f_! \simeq f_*$ commutes with base change along any map $S' \to S$, by noetherian induction, it is sufficient to prove this assertion after restricting to a dense open subscheme of S of our choice. Since, for h-motives, pulling back along a surjective étale morphism is conservative, we may even replace S by an étale neighbourhood of its generic points. For f surjective and radicial, [CD16, Proposition 6.3.16] ensures that $f_!$ is an equivalence of categories. We may thus assume that f also is étale. If ever $X = X' \amalg X''$, and if f' and f'' are the restriction of f to X' and X'', respectively, then we have $DM_h(X) \simeq DM_h(X') \times DM_h(X'')$, and the functor $f_!$ decomposes into

$$f_!(M) = f'_!(M') \oplus f''_!(M'')$$

for $M = (M', M'')$. Therefore, it is then sufficient to prove the proposition for f' and f'' separately. Replacing S by an étale neighbourhood of its generic points, we may thus assume that either X is empty, either f is an isomorphism, in which cases the assertion is trivial. □

2.4.7 Let $P(n)$ be the assertion that, whenever S is integral and $f : X \to Y$ is a separated morphism of S-schemes of finite type, such that the dimension of the generic fiber of X over S is smaller than or equal to n, then, for any locally construtible h-motive M on X, there is a dense open subscheme U of S such that the formation of $f_*(M)$ is compatible with base change along maps $S' \to U \subset S$.

From now on, we fix a separated morphism of S-schemes of finite type $f : X \to Y$; as well as a locally constructible h-motive M on X.

Lemma 2.4.8 *The property that there exists a dense open subscheme $U \subset S$ such that the formation of $f_*(M)$ is stable under any base change along maps $S' \to U \subset S$ is local on Y for the Zariski topology.*

Proof Indeed, assume that there is an open covering $Y = \bigcup_j V_i$ such that, for each j, there is a dense open subset $U_j \subset U$ with the property that the formation of the motive $(f^{-1}(V_j) \to V_j)_*(M_{f^{-1}(V_j)})$ is stable under any base change along maps of the form $S' \to U_j \subset S$. Since Y is noetherian, we may assume that there finitely many V_j's, so that $U = \bigcap_j U_j$ is a dense open subscheme of S. For any j, the formation of $(f^{-1}(V_j) \to V_j)_*(M_{f^{-1}(V_j)})$ is stable under any base change along maps of the form $S' \to U \subset S$. Since pulling back along open immersions commutes with any push-forward, one deduces easily that the formation of $f_*(M)$ is stable under any base change of the form $(f^{-1}(V_j) \to V_j)_*(M_{f^{-1}(V_j)})$ is stable under any base change along maps of the form $S' \to U \subset S$. □

Lemma 2.4.9 *Assume that there is a compactification of Y: an open immersion $j : Y \to \bar{Y}$ with $\bar{Y}$ a proper S-scheme. If there is a dense open subscheme U such that the formation of $(jf)_*(M)$ is compatible with all base changes along maps $S' \to U \subset S$, then the formation of $f_*(M)$ is compatible with all base changes along maps $S' \to U \subset S$.*

Proof This follows right away from the fact that pulling back along j is compatible with any base changes and from the fully faithfulness of the functor j_* (so that $j^* j_* f_*(M) \simeq f_*(M)$). □

Lemma 2.4.10 *Assume that S is integral, that the dimension of the generic fiber of X over S is $n \geq 0$, and that $P(n-1)$ holds. If X is smooth over S, and if M is dualizable, then there is a dense open subscheme of S such that the formation of $f_*(M)$ is stable under base change along maps $S' \to U \subset S$.*

Proof Since pulling back along open immersions commutes with any push-forward, and since Y is quasi-compact, the problem is local over Y. Therefore, we may assume that Y is affine. Let us choose a closed embedding $Y \subset \mathbf{A}^d_S$ determined by d functions $g_i : Y \to \mathbf{A}^1_S$, $1 \leq i \leq d$. For each index i, we may apply $P(n-1)$ to f, seen as open embedding of schemes over $\mathbf{A}^1_S$ through the structural map g_i. This provides a dense open subscheme U_i in $\mathbf{A}^1_S$ such that the formation of $f_*(M)$ is compatible with any base change of g_i along a map $S' \to \mathbf{A}^1_S$ which factors through U_i. Let V be the union of all the open subschemes $g_i^{-1}(U_i)$, $1 \leq i \leq d$, and let us write $j : V \to Y$ for the corresponding open immersion. Then the formation of $j_! j^* f_*(M)$ is compatible with any base change $S' \to S$. Let us choose a closed complement $i : T \to Y$ to j. Then T is finite: the reduced geometric fibers of T/S are traces on Y of the subvarieties of $\mathbf{A}^d$ determined by the vanishing of all the non constant polynomials $p_i(x_i) = 0$, $1 \leq i \leq d$, where $p_i(x)$ is a polynomial such that $U_i = \{p_i(x) \neq 0\}$.

We may now consider the closure $\bar{Y}$ of Y in $\mathbf{P}^n_S$. Any complement of V in $\bar{Y}$ is also finite over a dense open subscheme of S: the image in S of the complement of V in $\bar{V}$ is closed (since $\bar{V}$ is proper over S), and does not contain the generic point (since the generic fiber of X is not empty), so that we may replace S by the complement of this image. By virtue of Lemma 2.4.9, we may replace Y by $\bar{Y}$, so that we are reduced to the following situation: the scheme Y is proper over S, and there is a dense open immersion $j : V \to Y$ with the property that the formation of $j_! j^* f_*(M)$ is compatible with any base change $S' \to S$, and that after shrinking S, there is a closed complement $t : T \to Y$ of V which is finite over S. We thus have the following canonical cofiber sequence

$$j_! j^* f_*(M) \to f_*(M) \to i_* i^* f_*(M)$$

Let $p : Y \to S$ be the structural map (which is now proper). We already know that the formation of $j_! j^* f_*(M)$ is compatible with any base change of the form $S' \to S$. Therefore, it is sufficient to prove that, possibly after shrinking S, the formation of $i_* i^* f_*(M)$ has the same property. Since $i_! \simeq i_*$, this means that this is equivalent to the property that, possibly after shrinking S, the formation of $i^* f_*(M)$ is compatible with any base change of the form $S' \to S$. But the composed morphism pi being finite, by virtue of Proposition 2.4.6, we are reduced to prove this property for $p_* i_* i^* f_*(M)$. We then have the following canonical cofiber sequence

$$p_* j_! j^* f_*(M) \to (pf)_*(M) \to (pi)_* i^* f_*(M)$$

By virtue of Proposition 2.4.4, possibly after shrinking S, the formation of $(pf)_*(M)$ is compatible with any base change. Since p is proper, we have the proper base change formula (because $p_! \simeq p_*$), and therefore, the formation of $j_! j^* f_*(M)$ being compatible with any base change of the form $S' \to S$, the formation of $p_* j_! j^* f_*(M)$ is also compatible with any base change $S' \to S$. One deduces that, possibly after shrinking S the fomration of $(pi)_* i^* f_*(M)$ is also compatible with any base change $S' \to S$. □

Proof of Theorem 2.4.2 We observe easily that it is sufficient to prove the case where S is integral. We shall prove $P(n)$ by induction. The case $n = -1$ is clear. We may thus assume that $n \geq 0$ and that $P(n-1)$ holds true. Locally for the h-topology, radicial surjective and integral morphisms are isomorphisms; in particular, pulling back along a radicial surjective and integral morphism is an equivalence of categories which commutes with the six operations. There is a dense open subscheme U of S and a finite radicial and surjective map $U' \to U$, so that $X' = X \times_S U'$ has a dense open subscheme which is smooth over U' (it is sufficent to prove this over the spectrum of the field of functions of S, by standard limit arguments). Replacing S by U' and X by X', we may thus assume, without loss of generality, that the smooth locus of X over S is a dense open subscheme.

Let $j : V \to X$ be a dense open immersion such that V is smooth over S. Shrinking V, we may assume furthermore that $M_{|V}$ is dualizable in $DM_h(V)$. We choose a closed complement $i : Z \to X$ of V. With $N = i^!(M)$, we then have the following canonical cofiber sequence:

$$i_*(N) \to M \to j_* j^*(M)$$

By virtue of Lemma 2.4.10, possibly after shrinking S, we may assume that the formation of $j_*(M)$ is compatible with base changes along maps $S' \to S$. So is the formation of $i_*(N)$, since i is proper. Applying the functor f_* to the distinguished triangle above, we obtain the following cofiber sequence:

$$(fi)_*(N) \to f_*(M) \to (fj)_* j^*(M)\,.$$

We may apply Lemma 2.4.10 to fj and M, and observe that $P(n-1)$ applies to fi and N. Therefore, there exists a dense open subscheme $U \subset S$ such that the formation of $(fi)_*(N)$ and of $(fj)_* j^*(M)$ is compatible with any base change along maps $S' \to U \subset S$. This implies that the formation of $f_*(M)$ is compatible with such base changes as well. □

3 Characteristic classes

3.1 Künneth Formula

3.1.1 Let k be a field. All schemes will be assumed to be separated of finite type over k.

Theorem 3.1.2 *Let $f : X \to Y$ be a map of k-schemes, and T a k-scheme. Consider the square*

$$\begin{array}{ccc} T \times X & \xrightarrow{pr_2} & X \\ \downarrow{\scriptstyle 1\times f} & & \downarrow{\scriptstyle f} \\ T \times Y & \xrightarrow{pr_2} & Y \end{array}$$

obtained by multiplying $f : X \to Y$ and $T \to Spec(k)$. Then $pr_2^* f_* \cong (1\times f)_* pr_2^*$ holds.

Proof Since, for a field k and $S = Spec(k)$, the only dense open subscheme of S is S itself, the generic base change formula gives that the canonical map $pr_2^* f_*(M) \to (1 \times f)_* pr_2^*(M)$ is an isomorphism for any locally constructible motive M on X. Since we are comparing colimit preserving functors and since any motive is a colimit of locally constructible ones, this proves the theorem. □

Some consequences:

1. Take X, T to be k-schemes and $pr_2 : T \times X \to X$ the projection. Then, for any M locally constructible on X we have:

$$pr_2^* Hom(M, N) \cong Hom(pr_2^* M, pr_2^* N)\,.$$

   It is proved by producing a canonical map and then prove for a fixed N and reduce to the case where M is a generator, namely $M = f_\sharp \Lambda$ for smooth f. Then we get $Hom(M, N) \cong f_* f^* M$.
2. For a morphism $f : X \to Y$ consider the square below.

$$\begin{array}{ccc} T \times X & \xrightarrow{pr_2} & X \\ \downarrow{\scriptstyle 1\times f} & & \downarrow{\scriptstyle f} \\ T \times Y & \xrightarrow{pr_2} & Y \end{array}$$

   Then $pr_2^* f^! \cong (1 \times f)^! pr_2^*$.

For the proof observe that this is a local problem so that we may assume f is quasi-projective. The map f then has a factorization $f = g \circ i \circ j$ where g is smooth, i is a closed immersion, and j is an open immersion. Then $j^* = j^!$ and $g^* = g^!(-d)[-2d]$ so we reduce to the case where f is a closed immersion. Then f_* and $(1 \times f)_*$ are fully faithfull hence conservative. Therefore, it suffices to show

$$(1\times f)_* pr_2^* f^! \cong (1\times f)_*(1\times f)^! pr_2^* .$$

Since left hand side is isomorphic to

$$pr_2^* f_* f^! ,$$

we only need to commute $f_* f^!$ and pr_2^*. Now observe that $f_* f^!(M) \cong Hom(f_*\Lambda, M)$. So we deduce the commutation of $f_* f^!$ from the commutation with internal Hom and f_* (which we both know). We finally have proper basechange $pr_2^* f_*(\Lambda) \cong (1\times f)_* pr_2^*$ and this finishes the proof.

Remark 3.1.3 If f is smooth or M is 'smooth' (dualizable) then for all N we have

$$f^* Hom(M,N) \cong Hom(f^*M, f^*N)$$

(see Remark 2.1.2).

3.1.4 For X a k-scheme and $a : X \to Spec(k)$ we define the dualizing sheaf to be $I_X = a^!\Lambda$ and $\mathbf{D}_X = Hom(-, I_X)$. If X, Y are schemes we can consider their product $X\times Y$ with projections $p_X : X\times Y \to X$ and $p_Y : X\times Y \to Y$. If M, N are motivic sheaves on X, Y respectively, we can define

$$M \boxtimes N := p_X^* M \otimes p_Y^* N$$

and then, recalling that $A\otimes B \cong \mathbf{D}Hom(A, \mathbf{D}B)$, we get that

$$M\boxtimes N \cong \mathbf{D}(Hom(p_X^*M, \mathbf{D}p_Y^*N)) \cong \mathbf{D}Hom(p_X^*M, p_Y^!\mathbf{D}N)$$

and therefore

$$M\boxtimes \mathbf{D}N \cong \mathbf{D}Hom(p_X^*M, p_Y^!N)$$

Theorem 3.1.5 *Let X, Y be k-schemes and N locally constructible on Y. Then $p_Y^! N \cong I_X \boxtimes N$.*

Proof Let a_X and a_Y be the structure maps of X, Y to $Spec(k)$. Then

$$p_X^* a_X^! \cong p_Y^! a_Y^* .$$

We have $I_X = a_X^!(\Lambda)$ and $p_X^*(I_X) \cong p_X^!(\Lambda)$. Moreover:

$$p_X^*(I_X) \cong p_X^*(\mathbf{D}_X\Lambda) \cong \mathbf{D}_{X\times Y} p_X^!\Lambda \cong \mathbf{D}_{X\times Y} p_Y^*(I_Y) .$$

Then we have

$$\begin{aligned} p_X^* I_X \otimes p_Y^* N &\cong \mathbf{D} p_Y^* I_Y \otimes p_Y^* N \\ &\cong \mathbf{D} Hom(p_Y^* N, p_Y^* I_Y) \\ &\cong \mathbf{D} p_Y^* Hom(N, I_Y) \\ &\cong \mathbf{D} p_Y^* \mathbf{D} N \\ &\cong p_Y^! N \end{aligned}$$

Hence $I_X \boxtimes N \cong p_Y^! N$. □

Corollary 3.1.6 $I_X \boxtimes I_Y \cong I_{X \times Y}$.

Proof $I_{X\times Y} \cong p_Y^! a_Y^! \Lambda \cong p_Y^! I_Y \cong I_X \boxtimes I_Y$. □

Proposition 3.1.7 (Künneth Formula with compact support) *Let $f : U \to X$ and $g : V \to Y$ and let $M \in DM_h(U, \Lambda)$ and $N \in DM_h(V, \Lambda)$ then*

$$f_!(M) \boxtimes g_!(N) \cong (f \times g)_!(M \boxtimes N).$$

Proof Since $(f \times g)_! \cong (f \times 1)_!(1 \times g)_!$, we see that it is sufficient to prove this when f or g is the identity. Using the functorialities induced by permuting the factors $X \times Y \cong Y \times X$, we see that it is sufficient to prove the case where g is the identity. We then have a Cartesian square

$$\begin{array}{ccc} U \times Y & \xrightarrow{p_U} & U \\ {\scriptstyle f\times 1}\downarrow & & \downarrow{\scriptstyle f} \\ X \times Y & \xrightarrow{p_X} & X \end{array}$$

inducing an isomorphism

$$(f \times 1)_! p_U^* \cong p_X^* f_! .$$

The projection formula also gives

$$(f \times 1)_!(p_U^*(M)) \otimes p_Y^*(N) \cong (f \times 1)_!(M \boxtimes N)$$

so that we get $f_!(M) \boxtimes N \cong (f \times 1)_!(M \boxtimes N)$. □

Corollary 3.1.8 *For $X = Y$ we get $f_!(M) \otimes g_!(N) \cong \pi_! i^*(M \boxtimes N)$ where $\pi : U \times_X V \to X$ is the canonical map, while $i : U \times_X V \to U \times V$ is the inclusion map.*

Remark 3.1.9 For f, g proper we get $f_* M \boxtimes f_* N \cong (f \times g)_*(M \boxtimes N)$.

Theorem 3.1.10 *For $M \in DM_{h,lc}(X, \Lambda)$ and $N \in DM_{h,lc}(Y, \Lambda)$ we have*

$$\mathbf{D}(M \boxtimes N) \cong \mathbf{D}M \boxtimes \mathbf{D}N .$$

Proof We may assume that $M = f_*\Lambda$ and $N = g_*\Lambda$ with f, g proper. Then

$$
\begin{aligned}
\mathbf{D}f_*\Lambda \boxtimes \mathbf{D}g_*\Lambda &\cong f_*I_U \boxtimes g_*I_V \\
&\cong (f \times g)_*(I_U \boxtimes I_V) \\
&\cong (f \times g)_* I_{U\times V} \\
&\cong \mathbf{D}((f \times g)_*\Lambda) \\
&\cong \mathbf{D}(f_*\Lambda \boxtimes g_*\Lambda) \\
&\cong \mathbf{D}(M \boxtimes N).
\end{aligned}
$$

Hence $\mathbf{D}(M \boxtimes N) \cong \mathbf{D}M \boxtimes \mathbf{D}N$. □

Corollary 3.1.11 $\mathbf{D}M \boxtimes N \cong \mathcal{H}om(p_X^* M, p_Y^! N)$ *for M and N locally constructible.*

Corollary 3.1.12 (Künneth Formula in cohomology)

Let us consider $f : U \to X$ *and* $g : V \to Y$ *together with* $M \in DM_h(U, \Lambda)$ *and* $N \in DM_h(V, \Lambda)$. *Then*

$$f_*(M) \boxtimes g_*(N) \cong (f \times g)_*(M \boxtimes N).$$

Proof Functors of the form p_*, for p separated of finite type, commute with small colimits: since they are exact, it is sufficient to prove that they commute with small sums, which follows from [CD16, Prop. 5.5.10]. Therefore it is sufficient to prove this when M and N are (locally) constructible. In this case, the series of isomorphisms

$$
\begin{aligned}
f_*(M) \boxtimes g_*(N) &\cong \mathbf{D}\mathbf{D}(f_*(M) \boxtimes g_*(N)) \\
&\cong \mathbf{D}(\mathbf{D}f_*(M) \boxtimes \mathbf{D}g_*(N)) \\
&\cong \mathbf{D}(f_!\mathbf{D}M \boxtimes g_!\mathbf{D}N) \\
&\cong \mathbf{D}((f \times g)_!(\mathbf{D}M \boxtimes \mathbf{D}N)) \\
&\cong \mathbf{D}((f \times g)_!(\mathbf{D}(M \boxtimes N)) \\
&\cong \mathbf{D}\mathbf{D}((f \times g)_*(M \boxtimes N)) \\
&\cong (f \times g)_*(M \boxtimes N)
\end{aligned}
$$

proves the claim. □

Remark 3.1.13 In the situation of the previous corollary, if $X = Y = Spec(k)$, then also $X \times Y = Spec(k)$, so that the exterior tensor product $\boxtimes$ in $DM_h(X \times Y, \Lambda)$ simply corresponds to the usual tensor product $\otimes$ on $DM_h(k, \Lambda)$. We thus get a Künneth formula of the form

$$(a_U)_*(M) \otimes (a_V)_*(N) \cong (a_U \times a_V)_*(M \boxtimes N)\,.$$

Corollary 3.1.14 *Let us consider* $f : U \to X$ *and* $g : V \to Y$, *together with* $M \in DM_h(X, \Lambda)$ *and* $N \in DM_h(Y, \Lambda)$. *Then*

$$f^!(M) \boxtimes g^!(N) \cong (f \times g)^!(M \boxtimes N).$$

Proof For any separated morphism of finite type a, the functor $a^!$ commutes with small colimits (since, they are exact, it is sufficient to prove that they commutes with small sums, which is asserted by [CD16, Cor. 5.5.14]). It is thus sufficient to prove this formula for constructible motivic sheaves. Using the fact that the Verdier duality functor $\mathbf{D}$ exchanges $*$'s and !'s as well as Theorem 3.1.10, we see that it is sufficient to prove the analogous formula obtained by considering functors of the form $(f\times g)^*$ and f^*, g^*, which is obvious. □

Corollary 3.1.15 *Let X be a k-scheme together with $M, N \in DM_{h,lc}(X,\Lambda)$. If we denote by $\Delta : X \longrightarrow X \times X$ the diagonal map, then*

$$\Delta^!(\mathbf{D}M \boxtimes N) \cong Hom(M,N)\,.$$

We have indeed:

$$\begin{aligned}\Delta^!(\mathbf{D}M \boxtimes N) &\cong \mathbf{D}\Delta^*\mathbf{D}(\mathbf{D}M \boxtimes N)\\ &\cong \mathbf{D}\Delta^*(\mathbf{D}\mathbf{D}M \boxtimes \mathbf{D}N)\\ &\cong \mathbf{D}(M \otimes \mathbf{D}N)\\ &\cong Hom(M,N)\,.\end{aligned}$$

Corollary 3.1.16 *Let X and Y be k-schemes. We consider A, M in $DM_h(X,\Lambda)$ as well as B, N in $DM_h(Y,\Lambda)$, with both A and B locally constructible. Then there is a canonical isomorphism*

$$Hom(A,M) \boxtimes Hom(B,N) \cong Hom(A \boxtimes B, M \boxtimes N)\,.$$

Proof There is a canonical morphism

$$Hom(A,M) \boxtimes Hom(B,N) \longrightarrow Hom(A \boxtimes B, M \boxtimes N)$$

which is compatible with pull-backs along étale maps. We may thus work étale locally on X and Y and assume that both A and B are constructible. The family of motivic sheaves A and B for which this map is invertible being closed under colimits and Tate twists, we may assume, without loss of generality, that $A = f_!(\Lambda)$ and $B = g_!(\Lambda)$ for two separated morphisms of finite type $f : U \longrightarrow X$ and $g : V \longrightarrow Y$. We then get

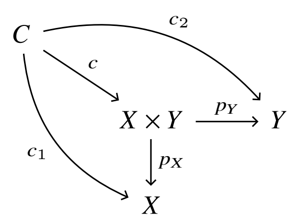
$$\begin{aligned}Hom(A,M) \boxtimes Hom(B,N) &= Hom(f_!(\Lambda),M) \boxtimes Hom(g_!(\Lambda),N)\\ &\cong f_*Hom(\Lambda, f^!(M)) \boxtimes g_*Hom(\Lambda, g^!(N))\\ &\cong f_*f^!(M) \boxtimes g_*g^!(N)\\ &\cong (f\times g)_*(f\times g)^!(M \boxtimes N)\\ &\cong Hom((f\times g)_!(\Lambda), M \boxtimes N)\\ &\cong Hom(f_!(\Lambda) \boxtimes g_!(\Lambda), M \boxtimes N)\,,\end{aligned}$$

where the fourth identification comes from Künneth Formulas 3.1.12 and 3.1.14, while the last one comes from Künneth Formula with compact support. □

3.2 Grothendieck-Lefschetz Formula.

As in the previous paragraph, we assume that a ground field k is given, and all schemes are assumed to be separated of finite type over k.

Definition 3.2.1 Let X and Y be schemes, together with $M \in DM_{h,lc}(X, \Lambda)$ and $N \in DM_{h,lc}(Y, \Lambda)$. A *cohomological correspondence* from (X, M) to (Y, N) is a triple of the form (C, c, α), where (C, c) determines the commutative diagram



together with a map $\alpha : c_1^* M \to c_2^! N$ in $DM_h(C, \Lambda)$.

Remark 3.2.2 We have:

$$Hom(c_1^* M, c_2^! N) \cong Hom(c^* p_X^* M, c^! p_Y^! N) \cong c^! Hom(p_X^* M, p_Y^! N) \cong c^!(\mathbf{D}M \boxtimes N).$$

Therefore, one can see α as a map of the form

$$\alpha : \Lambda \to c^!(\mathbf{D}M \boxtimes N)\,.$$

Remark 3.2.3 In the case where c_2 is proper, a cohomological correspondence induces a morphism in cohomology as follows. Let $a : X \to Spec(k)$ and $b : Y \to Spec(k)$ be the structural maps. We e have $ac_1 = bc_2$ and a co-unit map $(c_2)_* c_2^!(N) \to N$, whence a map:

$$a_* M \to a_*(c_1)_* c_1^* M \xrightarrow{a_*(c_1)_*\alpha} a_*(c_1)_* c_2^! N \cong b_*(c_2)_* c_2^! N \to b_* N\,.$$

In particular, one can consider the trace of such an induced map. By duality, in the case where c_1 is proper, we get an induced map in cohomology with compact support $b_! N \to a_! M$.

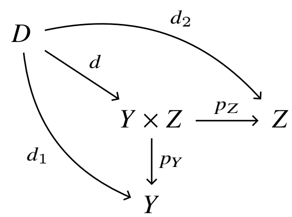
3.2.4 We observe that cohomological correspondences can be multiplied: given another cohomological correspondence (C', c', α') from (X', M') to (Y', N'), we define a new correspondence from $(X \times X', M \boxtimes M')$ to $(Y \times Y', N \boxtimes N')$ with

$$(C, c, \alpha) \otimes (C', c', \alpha') = (C \times C', c \times c', \alpha \boxtimes \alpha')$$

where $\alpha \boxtimes \alpha'$ is defined using the functoriality of the $\boxtimes$ operation together with the canonical Künneth isomorphisms seen in the previous paragraph:

$$\Lambda \cong \Lambda \boxtimes \Lambda \xrightarrow{\alpha\boxtimes\alpha'} c^!(\mathbf{D}M \boxtimes N) \boxtimes c'^!(\mathbf{D}M' \boxtimes N') \cong (c \times c')^!(\mathbf{D}(M \boxtimes M') \boxtimes (N \boxtimes N'))\,.$$

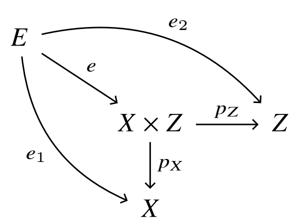
3.2.5 Correspondences can also be composed. Let (C, c, α) be a correspondence from (X, M) to (Y, N) as above, and let (D, d, β) be a correspondence from (Y, N) to (Z, P), with (D, d) corresponding to a commutative diagram of the form below, and $\beta : \Lambda \to d^!(\mathbf{D}N \boxtimes P)$ a map in in $DM_h(D, \Lambda)$.



We form the following pullback square

$$\begin{array}{ccc} E & \xrightarrow{\lambda} & D \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle d_1} \\ C & \xrightarrow{c_2} & Y \end{array}$$

as well as the commutative diagram



in which $e_1 = c_1\mu$ and $e_2 = d_2\lambda$. We then form $\alpha \boxtimes \beta$:

$$\Lambda \cong \Lambda \boxtimes \Lambda \xrightarrow{\alpha\boxtimes\beta} c^!(\mathbf{D}M \boxtimes N) \boxtimes d^!(\mathbf{D}N \boxtimes P) \cong (c \times d)^!((\mathbf{D}M \boxtimes N) \boxtimes (\mathbf{D}N \boxtimes P))\,.$$

Let $f = d_1\lambda = c_2\mu : E \to Y$ be the canonical map, and $\Delta : Y \to Y \times Y$ be the diagonal. We have the following Cartesian square

$$\begin{array}{ccc} E & \xrightarrow{(\mu,\lambda)} & C \times D \\ {\scriptstyle \varphi=(e_1,f,e_2)}\downarrow & & \downarrow{\scriptstyle c\times d} \\ X \times Y \times Z & \xrightarrow{1\times\Delta\times 1} & X \times Y \times Y \times Z \end{array}$$

which induces an isomorphism (proper base change formula)

$$\varphi_!(\mu,\lambda)^* \cong (1\times\Delta\times 1)^*(c\times d)_!\,.$$

In particular, it induces a canonical map

$$\kappa : (\mu,\lambda)^*(c\times d)^! \to \varphi^!(1\times\Delta\times 1)^*$$

corresponding by adjunction to the composite

$$\varphi_!(\mu,\lambda)^*(c\times d)^! \cong (1\times\Delta\times 1)^*(c\times d)_!(c\times d)^! \xrightarrow{\text{co-unit}} (1\times\Delta\times 1)^*\,.$$

Let $\pi : X\times Y\times Z \to X\times Z$ be the canonical projection. There is a canonical map

$$\varepsilon : (1\times\Delta\times 1)^*(\mathbf{D}M \boxtimes N \boxtimes \mathbf{D}N \boxtimes P) \to \pi^!(\mathbf{D}M\boxtimes P)$$

induced by the evaluation map

$$N\otimes \mathbf{D}N \to I_Y$$

together with the canonical identifications coming from appropriate Künneth formulas:

$$\begin{aligned}(1\times\Delta\times 1)^*(\mathbf{D}M \boxtimes (N \boxtimes \mathbf{D}N) \boxtimes P) &\cong \mathbf{D}M \boxtimes (N\otimes \mathbf{D}N)\boxtimes P\\ \mathbf{D}M\boxtimes I_Y\boxtimes P &\cong \pi^!(\mathbf{D}M\boxtimes P)\,.\end{aligned}$$

We observe that $e = \pi\varphi$, so that $e^! \cong \varphi^!\pi^!$.

Definition 3.2.6 With the notations above, composing $(\mu,\lambda)^*(\alpha\boxtimes\beta)$ with the maps κ and ε defines the map

$$\beta\circ\alpha : \Lambda \cong (\mu,\lambda)^*\Lambda \to \varphi^!\pi^!(\mathbf{D}M\boxtimes P) \cong e^!(\mathbf{D}M\boxtimes P)\,.$$

We define finally define the *composition of the correspondences* (C,c,α) and (D,d,β) as

$$(D,d,\beta)\circ(C,c,\alpha) = (E,e,\beta\circ\alpha)\,.$$

3.2.7 This composition is only well defined up to isomorphism (since some choice of pull-back appears), but it is associative and unital up to isomorphism. The unit cohomological correspondence of (X,M) is given by

$$1_{(X,M)} = (X,\Delta,1_M)$$

where $\Delta : X \to X\times X$ is the diagonal map and

$$1_M : \Lambda \to \Delta^!(\mathbf{D}M\boxtimes M) \cong \mathit{Hom}(M,M)$$

is the canonical unit map. In a suitable sense, this defines a symmetric monoidal bicategory, where the tensor product is defined as

$$(X, M) \otimes (Y, N) = (X \times Y, M \boxtimes N)$$

while the unit object if $(Spec(k), \Lambda)$.

3.2.8 To make this a little bit more precise, we must speak of the category of cohomological correspondences from (X, M) to (Y, N), in order to be able to express the fact that all the contructions and all the coherence isomorphisms (expressing the associativity and so on) are functorial. If (C, c, α) and (D, d, β) both are correspondences from (X, M) to (Y, N), a map

$$\sigma : (C, c, \alpha) \to (D, d, \beta)$$

is a pair $\sigma = (f, h)$, where $f : C \to D$ is a proper morphism such that $df = c$, while h is a homotopy

$$h : f_!(\alpha) \cong \beta$$

where $f_!(\alpha)$ is the map defined as

$$f_!(\alpha) : \Lambda \xrightarrow{\text{unit}} f_*\Lambda \xrightarrow{f_*\alpha} f_*c^!(\mathbf{D}M \boxtimes N) \cong f_*f^!d^!(\mathbf{D}M \boxtimes N) \xrightarrow{\text{co-unit}} d^!(\mathbf{D}M \boxtimes N).$$

This defines the symmetric monoidal bicategory $MCorr(k)$ whose objects are the pairs (X, M) formed of a k-scheme X equipped with a Λ-linear locally constructible h-motive M. In particular, for each pair of pairs (X, M) and (Y, N), there is the category of cohomological correspondences from (X, M) to (Y, N), denoted by $MCorr(X, M; Y, N)$ (in this paragraph, unless we make it explicit otherwise, we will only need the 1-category of such things, considering maps α as above in the homotopy category of h-motives).

Proposition 3.2.9 *All the objects of $MCorr(k)$ are dualizable. Moreover, the dual of a pair (X, M) is $(X, \mathbf{D}M)$.*

Proof Let (X, M), (Y, N) and (Z, P) be three objects of $MCorr(k)$. A cohomological correspondence from $(X \times Y, M \boxtimes N)$ to (Z, P) is determined by a morphism of k-schemes $c : C \to X \times Y \times Z$ together with a map

$$\alpha : \Lambda \to c^!(\mathbf{D}(M \boxtimes N) \boxtimes P)\,.$$

A cohomological correspondence from (X, M) to $(Y \times Z, \mathbf{D}N \boxtimes P)$ is determined by a morphism of k-schemes $c : C \to X \times Y \times Z$ together with a map

$$\alpha : \Lambda \to c^!(\mathbf{D}M \boxtimes (\mathbf{D}N \boxtimes P))\,.$$

The Künneth formula

$$\mathbf{D}(M \boxtimes N) \boxtimes P \cong \mathbf{D}M \boxtimes (\mathbf{D}N \boxtimes P)$$

implies our assertion. □

3.2.10 Let X be a scheme and M a locally constructible h-motive on X. We denote by $\Delta : X \to X \times X$ the diagonal map. There is a *transposed evaluation map*

$$ev^t_M : \mathbf{D}M \boxtimes M \to \Delta_* I_X$$

which corresponds by adjunction to the classical evaluation map

$$\Delta^*(\mathbf{D}M \boxtimes M) \cong \mathcal{H}om(M, I_X) \otimes M \to I_X \,.$$

Definition 3.2.11 Let (C, c, α) be a cohomological correspondence from (X, M) to (Y, N). In the case $(X, M) = (Y, N)$ we can form the following Cartesian square.

$$\begin{array}{ccc} F & \xrightarrow{\delta} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle c} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

The scheme F is called the *fixed locus* of the correspondence (C, c). The transposed evaluation map of M induces by proper base change a map

$$c^!(ev^t_M) : c^!(\mathbf{D}M \boxtimes M) \to c^!\Delta_* I_X \cong \delta_* p^! I_X \cong \delta_* I_F \,,$$

and thus, by adjunction, a map

$$ev^t_{M,c} : \delta^* c^!(\mathbf{D}M \boxtimes M) \to I_F \,.$$

The map $\alpha : \Lambda \to c^!(\mathbf{D}M \boxtimes M)$ finally induces a map

$$Tr(\alpha) : \Lambda \cong p^*\Lambda \to I_F$$

defined as the composition of $\delta^*\alpha$ with $ev^t_{M,c}$ (modulo the identification $\delta^*\Lambda \cong \Lambda$). The corresponding class

$$Tr(\alpha) \in H^0 \mathrm{Hom}_{DM_h(F,\Lambda)}(\Lambda, I_F)$$

is called the *characteristic class* of α.

Example 3.2.12 Let $f : X \to X$ be a morphism of schemes, and let M be a Λ-linear locally constructible h-motive on X, equipped with a map $\alpha : f^*M \to M$. Then $(X, (1_X, f), \mathbf{D}\alpha)$ is a cohomological correspondence from $(X, \mathbf{D}M)$ to itself, with

$$\mathbf{D}\alpha : 1_X^* \mathbf{D}M \cong \mathbf{D}M \to \mathbf{D}f^*M \cong f^! \mathbf{D}M \,.$$

If we form the Cartesian square

$$\begin{array}{ccc} F & \xrightarrow{\delta} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (1_X, f)} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

we see that F is indeed the fixed locus of the morphism f. If $\Lambda \subset \mathbf{Q}$, then he associated characteristic class

$$Tr(\mathbf{D}\alpha) \in H^0 \mathrm{Hom}(\Lambda, I_F) \otimes \mathbf{Q} \cong CH_0(F) \otimes \mathbf{Q}$$

defines a 0-cycle on F (see Theorem 1.4.3). In the case where f only has isolated fixed points, we have

$$CH_0(F) \otimes \mathbf{Q} \cong CH_0(F_{red}) \otimes \mathbf{Q} \cong \oplus_{i \in I} CH_0(Spec(k_i)) \otimes \mathbf{Q}$$

where I is a finite set and each k_i is a finite field extension of k with $F_{red} = \coprod_i Spec(k_i)$. Using this decomposition, one can then express the characteristic class of α as a sum of local terms: the contributions of each summand $CH_0(Spec(k_i)) \otimes \mathbf{Q}$. For instance, if U is an open subset of X such that $f(U) \subset U$, and if $j : U \to X$ is the inclusion map, we can consider $M = j_! \Lambda$ and the canonical isomorphism $\alpha : f^* j_! \Lambda \to j_! \Lambda$, in which case $Tr(\alpha)$ is a way to count the number of fixed points of f in U with 'arithmetic multiplicities' (in the form of 0-cycles).

Remark 3.2.13 The notation $Tr(\alpha)$ is justified by Proposition 3.2.9: indeed, essentially by definition of the composition law for cohomological correspondences sketched in paragraph 3.2.6, the characteristic class $Tr(\alpha)$ is the trace of the endomorphism (C, c, α) of the dualizable object (X, M). Indeed, the endomorphisms of $(Spec(k), \Lambda)$ in $MCorr(k)$ are determined by pairs (F, t) where F is a k-scheme and $t : \Lambda \to I_F$ is a section of the dualizing object of F in $DM_h(F, \Lambda)$.

Corollary 3.2.14 *For any cohomological correspondences (C, c, α) and (D, d, β) from (X, M) to itself, we have:*

$$Tr(\beta \circ \alpha) = Tr(\alpha \circ \beta)\,.$$

Corollary 3.2.15 *Let (C, c, α) be a cohomological correspondence from (X, M) to itself. If we see α as a map from $c_1^* M \to c_2^! M$, it determines a map*

$$\mathbf{D}\alpha : c_2^* \mathbf{D}M \cong \mathbf{D} c_2^! M \to \mathbf{D} c_1^* M \cong c_1^! \mathbf{D}M\,.$$

If $\tau : X \times X \to X \times X$ denotes the permutation of factors, the cohomological correspondence $(C, \tau c, \mathbf{D}\alpha)$ from $(X, \mathbf{D}M)$ to itself is the explicit description of the map obtained from (C, c, α) by duality. In particular:

$$Tr(\alpha) = Tr(\mathbf{D}\alpha)\,.$$

3.2.16 The formation of traces is functorial with respect to morphisms of correspondences. Let M be a locally constructible motive on a scheme X, and $f : C \to D$, $d : D \to X \times X$, $c = df$, be morphisms, with f proper. We form pull-back squares

$$\begin{array}{ccc} F & \xrightarrow{\delta} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\varepsilon} & D \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle d} \\ X & \xrightarrow{\Delta} & X \times X \end{array} \qquad p = qg : F \to X, \quad c = df : C \to X \times X$$

and have a composition

$$f_* c^! (\mathbf{D}M \boxtimes N) \cong f_* f^! d^! (\mathbf{D}M \boxtimes N) \xrightarrow{\text{co-unit}} d^! (\mathbf{D}M \boxtimes N)$$

as well as a composition

$$f_* \delta_* I_F \cong \varepsilon_* g_* I_F \cong \varepsilon_* g_* g^! I_G \xrightarrow{\text{co-unit}} \varepsilon_* I_G \,.$$

One then checks right away that the following square commutes.

$$\begin{array}{ccc} f_* c^! (\mathbf{D}M \boxtimes N) & \longrightarrow & d^! (\mathbf{D}M \boxtimes N) \\ {\scriptstyle f_* c^! (ev^t_M)}\downarrow & & \downarrow{\scriptstyle g_* d^! (ev^t_M)} \\ f_* \delta_* I_F & \longrightarrow & \varepsilon_* I_G \end{array}$$

This implies immediately that, for any map$\alpha : \Lambda \to c^! (\mathbf{D}M \boxtimes M)$, we have:

$$Tr(\alpha) = Tr(f_!(\alpha)) \,.$$

3.2.17 Proper maps act on cohomological correspondences as follows. We consider a proper morphism of geometric correspondences, by which we mean a commutative square of the form

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ {\scriptstyle c=(c_1,c_2)}\downarrow & & \downarrow{\scriptstyle d=(d_1,d_2)} \\ X \times X' & \xrightarrow{f \times f'} & Y \times Y' \end{array}$$

in which $f : X \to Y$, $f' : X' \to Y'$ and $\varphi : C \to D$ are proper map, together with locally constructible h-motives M on X and M' on X'. Given a cohomological correspondence from (X, M) to (X', M') of the form (C, c, α), we have a cohomological correspondence from $(X, f_! M)$ to $(X', f'_! M')$

$$(f, f')_! (C, c, \alpha) = (C, d\varphi, (f, f')_! (\alpha))$$

defined as follows. If, furthermore, the commutative square above is Cartesian, the map $(f, f')_! (\alpha)$ is the induced map

$$\Lambda \xrightarrow{\text{unit}} \varphi_* \Lambda \xrightarrow{\varphi_* \alpha} \varphi_* c^! (\mathbf{D}M \boxtimes M') \cong d^! (f \times f')_* (\mathbf{D}M \boxtimes M') \cong d^! (\mathbf{D} f_! M \boxtimes f'_! M')$$

Otherwise, we consider the induced proper map

$$g : C \to E = X \times X' \times_{Y \times Y'} D$$

and apply the preceding construction to $g_!(\alpha)$, replacing C by E.

In the case where $(X, M) = (X', M')$ and $f = f'$, we simply write

$$f_!(\alpha) = (f, f)_!(\alpha) .$$

Theorem 3.2.18 (Lefschetz-Verdier Formula) *We consider a commutative square of k-schemes of finite type of the form*

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ {\scriptstyle c=(c_1,c_2)}\downarrow & & \downarrow{\scriptstyle d=(d_1,d_2)} \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

in which both f and φ are proper, as well as a locally constructible h-motive M on X, together with a map $\alpha : \Lambda \to c^!(\mathbf{D}M \boxtimes M)$. Let F and G be the fixed locus of (C, c) and (D, d) respectively. Then the induced map $\psi : F \to G$ is also proper, and

$$\psi_!(Tr(\alpha)) = Tr(f_!(\alpha)) .$$

Proof The functoriality of the trace explained in 3.2.16 shows that it is sufficient to prove the theorem in the case where the square is Cartesian. We check that the two maps

$$(f \times f)_*(\mathbf{D}M \boxtimes M) \cong (\mathbf{D}f_!M \boxtimes f_!M) \xrightarrow{ev^t_{f_!M}} \Delta_* I_Y$$

and

$$(f \times f)_*(\mathbf{D}M \boxtimes M) \xrightarrow{(f \times f)_*(ev^t_M)} (f \times f)_*\Delta_* I_X \cong \Delta_* f_* I_X \cong \Delta_* f_! f^! I_Y \xrightarrow{\text{co-unit}} \Delta_* I_Y$$

are equal (where we have denoted by the same symbol the diagonal of X and the diagonal of Y). By duality, this amounts to check that the unit map

$$\Delta_*\Lambda \to M \boxtimes \mathbf{D}M$$

is compatible with the push-forward f_*. This is a fancy way to say that f_*M has a natural $f_*\Lambda$-algebra structure, which comes from the fact that the functor f^* is symmetric monoidal. The Lefschetz-Verdier Formula follows then right away. □

Remark 3.2.19 When $\Lambda = \mathbf{Q}$, the operator $\psi_!$ coincides with the usual push-forward of 0-cycles: seen as a map

$$\psi_! : H^0\mathrm{Hom}(\Lambda, I_F) \to H^0\mathrm{Hom}(\Lambda, I_G) .$$

Theorem 3.2.20 (Additivity of Traces) *Let $c = (c_1, c_2) : C \to X \times X$ be a correspondence of k-schemes. We consider a cofiber sequence*

$$M' \to M \to M''$$

in $DM_{h,lc}(X)$ *as well as maps*

$$\alpha' : c_1^* M' \to c_2^! M' \,,\ \alpha : c_1^* M \to c_2^! M \,,\ \alpha'' : c_1^* M'' \to c_2^! M''$$

in $DM_{h,lc}(C)$ *so that the diagram below commutes (in the sense of* ∞*-categories).*

$$\begin{array}{ccccc} c_1^* M' & \longrightarrow & c_1^* M & \longrightarrow & c_1^* M'' \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} \\ c_2^! M' & \longrightarrow & c_2^! M & \longrightarrow & c_2^! M'' \end{array}$$

Then the following formula holds.

$$Tr(\alpha) = Tr(\alpha') + Tr(\alpha'')$$

The proof is given in the paper of Jin and Yang [JY18, Theorem 4.2.8] using the language of algebraic derivators, which is sufficient for our purpose (note however that, by Balzin's work [Bal19, Theorem 2], it is clear that one can go back and forth between the language of fibred ∞-categories and the one of algebraic derivators). The additivity of traces can be extended to more general homotopy colimits; see Gallauer's thesis [Gal14].

Remark 3.2.21 It is pleasant to observe that, when $\Lambda = \mathbf{Q}$, this is the classical push-forward of 0-cycles. Lefschetz-Verdier Formula is particularly relevant in the case where $Y = Spec(k)$, and F consists of isolated points in X (in which case it is called the Grothendieck-Lefschetz formula). Indeed, $f_!(M)$ is then the cohomology of X with compact support with coefficients in M, so that $Tr(f_!(\alpha))$ is the ordinary trace of the endomorphism $f_!(\alpha) : f_!(M) \to f_!(M)$, which can be computed through ℓ-adic realizations as an alternating sum of ordinary traces of linear maps. On the other hand, $\psi_!(Tr(\alpha))$ is the sum of traces of the endomorphisms induced by α on each $p_! x^* M$, where x runs over the points of F, with $p : Spec(\kappa(x)) \to Spec(k)$ the structural map. In the particular case discussed at the end of Example 3.2.12, this shows that one can compute the number of fixed points with geometric multiplicities of a endomorphism of a k-scheme $f : X \to X$ with isolated fixed points which extends to an endomorphism of a compactification of X and whose graph is transverse to the diagonal, using the trace of the induced endomorphism of the motive with compact support of X. For the Frobenius map, such an extension is automatic, so that We can count rational points of any separated $\mathbf{F}_q$-scheme of finite type X_0 over a finite field $\mathbf{F}_q$ with the Grothendieck-Lefschetz formula

$$\# X(\mathbf{F}_q) = \sum_i (-1)^i \operatorname{Tr}\big(F : H^i_c(X, \mathbf{Q}_\ell) \to H^i_c(X, \mathbf{Q}_\ell)\big)\,,$$

where X is the pull-back of X_0 on the algebraic closure $\mathbf{F}_q$, and where F is a the map induced by the geometric Frobenius (i.e. where one considers the correspondence defined by the transposed graph of the arithmetic Frobenius). Indeed, using the additivity of traces, it is in fact sufficient to prove this formula in the case where X is smooth and projective, in which case the classical Lefschetz formula applies. We will now prove a more general version of it: we will consider arbitrary (locally) constructible motivic sheaves as coefficients.

3.2.22 Let p be a prime number, $r > 0$ a natural number, and $q = p^r$. Let $k_0 = \mathbf{F}_q$ be the finite field with q elements, and let us choose an algebraic closure k of k_0. Given a $\mathbf{F}_p$-scheme X, we denote by

$$F_X : X \to X$$

the *absolute Frobenius* of X, given by the identity on the underlying topological space, and by $a \mapsto a^p$ on the structural sheaf $\mathcal{O}_X$. The absolute Frobenius is a natural transformation from the identity of the category of $\mathbf{F}_p$-schemes to itself. In particular, for any morphism of k-schemes $u : U \to X$, there is a commutative square

$$\begin{array}{ccc} U & \xrightarrow{F_U} & U \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle u} \\ X & \xrightarrow{F_X} & X \end{array}$$

and thus a comparison map:

$$F_{U/X} = (u, F_U)\colon U \to F_X^{-1}(U) = X \times_X U$$

called the *relative Frobenius* of U over X. In case X_0 is a k_0-scheme, the rth iteration of the absolute Frobenius

$$F^r_{X_0}\colon X_0 \to X_0$$

is often called the *q-absolute Frobenius of* X_0 (and has the feature of being a map of k_0-schemes). By base change to k, it induces the *geometric Frobenius of* X, i.e. the morphism of k-schemes

$$\phi_r : X \to X\,,$$

where $X = \operatorname{Spec}(k) \times_{\operatorname{Spec}(k_0)} X_0$. Following Deligne's conventions, sheaves (or motives) on X_0 will often be denoted by M_0, and the pullback of M_0 along the canonical projection $X \to X_0$ will be written M. The map $k \to k$, defined by $x \mapsto x^q$ is an automorphism of k_0-algebras, which induces an isomorphism of k_0-schemes

$$Frob_q\colon \operatorname{Spec}(k) \to \operatorname{Spec}(k)\,.$$

It induces an isomorphism of k_0-schemes

$$Frob_{q,X} = (Frob_q \times_{\operatorname{Spec}(k_0)} 1_{X_0})\colon X \to X$$

whose composition with ϕ_r is nothing else than the absolute Frobenius of X. The map

$$Frob^{-1}_{q,X}\colon X \to X$$

is often called the *arithmetic Frobenius of X*.

Lemma 3.2.23 *Let X be a locally noetherian $\mathbf{F}_p$-scheme. The functor*

$$F_X^* : DM_h(X,\Lambda) \to DM_h(X,\Lambda)$$

is the identity.

Proof Let $a : X \to \mathrm{Spec}(\mathbf{F}_p)$ be the structural map. We have a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\ \mathrm{Spec}(\mathbf{F}_p) & = & \mathrm{Spec}(\mathbf{F}_p) \end{array}$$

in which the map F_X is a universal homeomorphism (being integral, radicial and surjective) and thus invertible locally for the h-topology. In other words, the square above is Cartesian locally for the h-topology. By h-descent, the functor

$$F_X^* : DM_h(X,\Lambda) \to DM_h(X,\Lambda)$$

thus acts as the identity. □

Remark 3.2.24 For a k_0-scheme X_0, since the composition of the geometric Frobenius $\phi_r : X \to X$ with the inverse of the arithmetic Frobenius is the absolute Frobenius, this shows that considering actions of the geometric Frobenius or of the arithmetic Frobenius amount to the same thing, at least as far as motivic sheaves are concerned. In fact the previous lemma is also a way to define such actions.

Let M_0 be a motivic sheaf on X_0, i.e. an object of $DM_h(X_0,\Lambda)$. Since $F_X = Frob_{q,X}\,\phi_r$ we have

$$F_X^*(M) = M \simeq \phi_r^*\, Frob_{q,X}^*(M)\,.$$

On the other hand, since M_0 is defined over k_0, and $M = a^*(M_0)$, there is a canonical isomorphism

$$Frob_{q,X}^*(M) \cong M\,.$$

Therefore, we have a canonical isomorphism

$$\phi_r^*(M) \cong M = (1_X)^!(M)\,.$$

Since the locus of fixed points of ϕ_r is precisely the (finite) set $X(k_0)$ of rational points of X_0 (seen as a discrete algebraic variety over k), the Verdier trace of the isomorphism above defines a class

$$L(M_0) = Tr\Big(\phi_r^*(M) \xrightarrow{\cong} (1_X)^!(M)\Big) \in H_0(X(k_0),\Lambda)\,.$$

If p is invertible in Λ, we have simply

$$H_0(X(k_0),\Lambda) \cong \Lambda^{X(k_0)}\,.$$

Under this identification, the obvious function

$$\Lambda^{X(k_0)} \to \Lambda\,, \quad f \mapsto \int f = \sum_{x \in X(k_0)} f(x)$$

coincides with the operator

$$\psi_! : H_0(X(k_0), \Lambda) \to H_0(\mathrm{Spec}(k), \Lambda) = \Lambda$$

induced by the structural map $\psi : X(k_0) \to \mathrm{Spec}(k)$.

The action of Frobenius is functorial: for any map $f : X_0 \to Y_0$, the induced action of ϕ_r^* on $f_!(M)$ via the proper base change isomorphism

$$\phi_r^* f_! \cong f_! \phi_r^*$$

coincides with the action defined as above in the case of $f_!(M_0)$. There is a similar compatibility with the canonical isomorphism $\phi_r^* f^* \cong f^* \phi_r^*$.

If X_0 is proper and if $j : U_0 \to X_0$ is an open immersion, for any M_0 locally constructible in $DM_h(U_0, \mathbf{Q})$, we thus get, as a special case of the Lefschetz-Verdier Formula (Theorem 3.2.18)

$$Tr\Big(\phi_r^* : a_!(M) \to a_!(M)\Big) = \int L(j_! M_0) \in \mathbf{Q}$$

where $a : U \to \mathrm{Spec}(k)$ is the structural morphism.

Theorem 3.2.25 (Grothendieck-Lefschetz Formula) *Let $j : U_0 \to X_0$ is an open immersion into a proper scheme of finite type over a finite field k_0 and let M_0 be a locally constructible motivic sheaf in $DM_h(U_0, \mathbf{Q})$. For each rational point x of U_0, we denote by M_x the fiber of M at the induced geometric point of U, on which there is a canonical action of the geometric Frobenius (as a particular case of the construction of Remark 3.2.24). Then*

$$Tr\Big(\phi_r^* : a_!(M) \to a_!(M)\Big) = \sum_{x \in U(k_0)} Tr\Big(\phi_r^* : M_x \to M_x\Big)\,.$$

Proof The case where $M_0 = \mathbf{Q}$ is constant is well known (see Remark 3.2.21). This proves the case where $M_0 = p_!(\mathbf{Q})$ for a map $p : Y_0 \to U_0$. The case of a direct factor of $p_!(\mathbf{Q})$ with Y_0 smooth and projective can be proved in the same way: the projector defining our motive is then given by some $\dim(Y)$-dimensional cycle α on $Y \times Y$ supported on $Y \times_X Y$ (see Theorem 1.4.3). We then observe that the Grothendieck-Lefschetz fixed point formula holds (using proper base change formula and Olsson's computation of local terms [Ols15, Prop. 5.5]). On the other hand, we see that the shift $[i]$ and the Tate twist (n) are compatible with traces (they consist in multiplying by $(-1)^i$ and by 1, respectively). By the additivity of traces, we are comparing two numbers which only depend on the class of M_0 in the Grothendieck group $K_0(DM_{h,lc}(U_0, \mathbf{Q}))$, and it is sufficient to consider the case where $U_0 = X_0$ is projective. Using Bondarko's theory of motivic weights [Bon14, Prop. 3.3], we see that any class in $K_0(DM_{h,lc}(U_0, \mathbf{Q}))$ is a linear combination of classes of motives which are direct factors of $p_*(\mathbf{Q})(n)[i]$ for $n, i \in \mathbf{Z}$ and $p : Y_0 \to U_0$ a projective morphism, with Y_0 smooth and projective. This proves the formula in general. □

Remark 3.2.26 When $M_0 = p_!(\mathbf{Q})$, with $p : Y_0 \to U_0$ separated of finite type, the Grothendieck-Lefschetz Formula expresses the trace of the Frobenius action on cohomology with compact support of Y as a sum of the traces of the action of Frobenius on cohomology with compact support of the fibers Y_x of Y over each k_0-rational point x of U. One can do similar constructions replacing the geometric Frobenius action by any (functorially given) automorphism of k-schemes, such as the identity. The computation of the local terms given by the Lefschetz-Verdier Trace Formula can then be rather involved. For instance, in the case of the identity (which means that we want to compute Euler-Poincaré characteristic of cohomology with compact support), the naive formula tends to fail (at least in positive characteristic). The Grothendieck-Ogg-Shafarevitch Formula is such a non-trivial computation in the case where M_0 is dualizable on a smooth curve U_0: it measures the deffect of the naive formula in terms of Swan conductors.

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